1. Red curves are loci of the stable fixed points, green curves the unstable ones.

   (a) $x_e = \frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 - 1}$. Saddle node bifurcations at $a = a_c = \pm 2$.

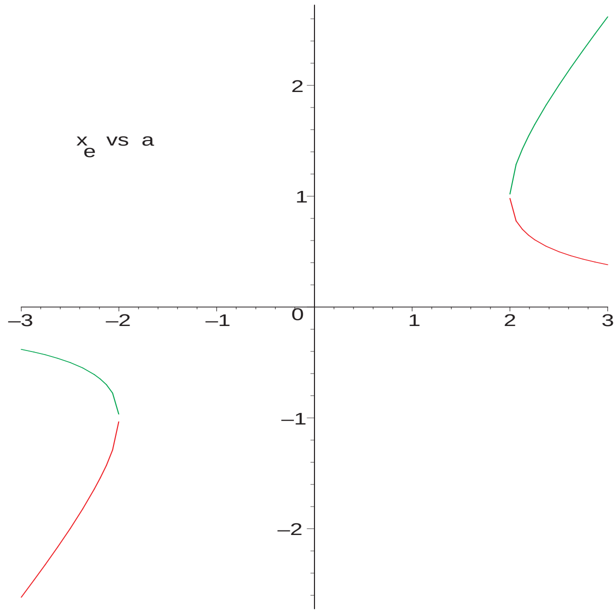

Figure 1: Bifurcation Diagram for Q1 (a)

   (b) One solution of $a + x_e = e^{-x_e}$. Also $f'(x_e) = 1 + e^{-x_e} > 0$, so the fixed point is unstable for all a.

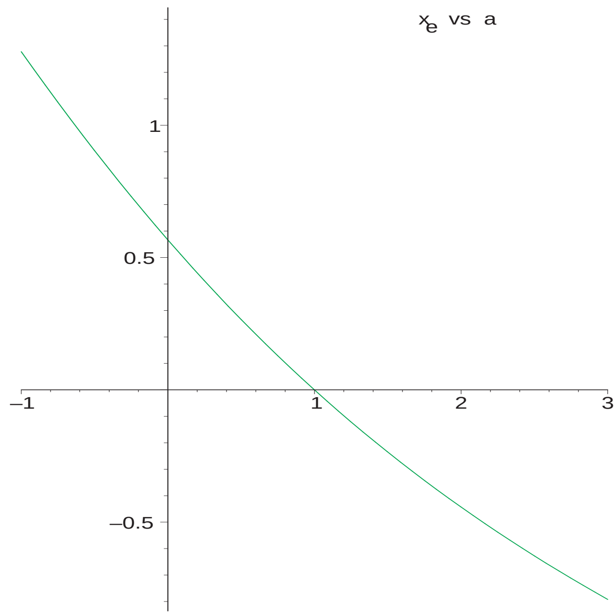

Figure 2: Bifurcation Diagram for Q1 (b)

   (c) $x_e = 0,\ 1 - 1/a$ and $f'(x_e) = 1 - a,\ a - 1$ respectively. Transcritical bifurcation at $a_c = 1$.

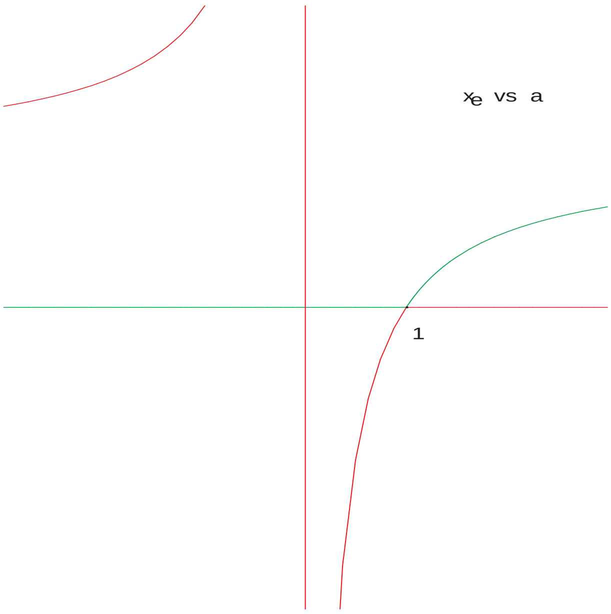

Figure 3: Bifurcation Diagram for Q1 (c)

(d) $x_e = 0, \ \pm\sqrt{-(1+a)}$ and $f'(x_e) = 1 + a, \ -(1+a)$ respectively. Subcritical pitchfork bifurcation at $a_c = -1$.

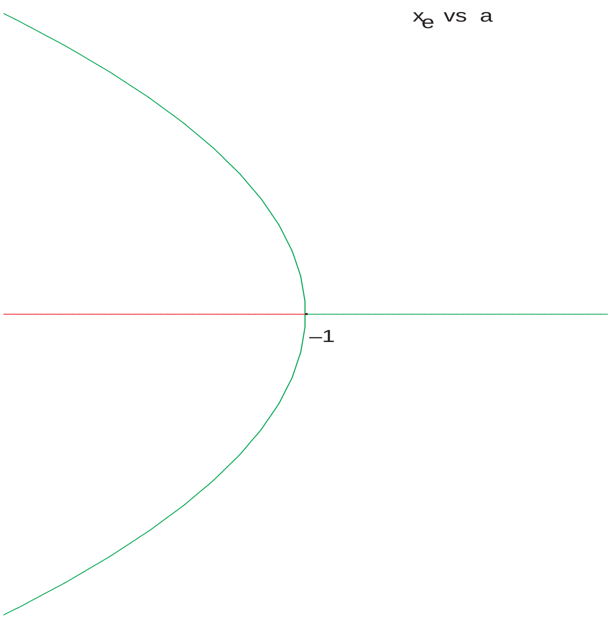

Figure 4: Bifurcation Diagram for Q1 (d)

(e) $x_e = 0, \ \pm\sqrt{1/a - 1}$ and $f'(x_e) = a - 1, \ 2a(1-a)$ respectively. Subcritical pitchfork bifurcation at $a_c = 1$. In addition there is a saddle node bifurcation at $a_c = 0$ with hysteresis occurring at this value of a.

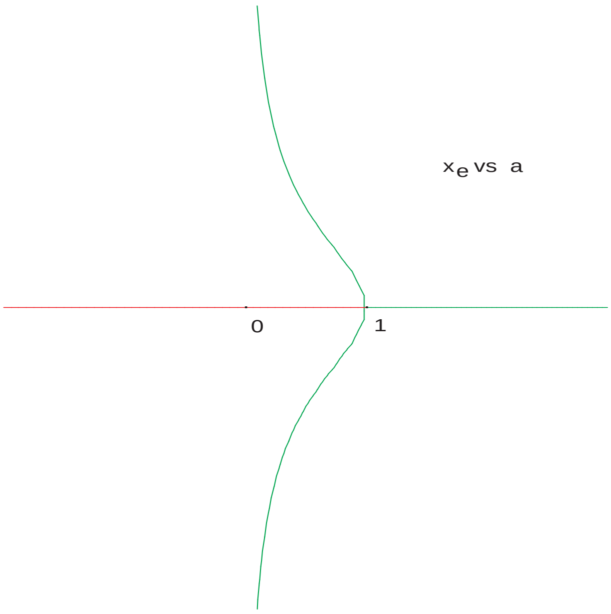

Figure 5: Bifurcation Diagram for Q1 (e)

2. (a) Steps are (i) divide by K, (ii) replace N/K by x, (iii) let $\tau = st \Rightarrow \frac{d(.)}{d\tau} = \frac{1}{s}\frac{d(.)}{dt}$. (iv) Choose $s = r$ and let $h = H/rK$, i.e.

$$\begin{aligned}
\dot{N} &= rN(1-\frac{N}{K}) - H \\
\Rightarrow \left(\frac{\dot{N}}{K}\right) &= r\frac{N}{K}(1-\frac{N}{K}) - \frac{H}{K} \\
\Rightarrow \dot{x} &= rx(1-x) - \frac{H}{K} \\
\Rightarrow \frac{dx}{d\tau} &= \frac{r}{s}x(1-x) - \frac{H}{sK} \\
\Rightarrow \frac{dx}{d\tau} &= x(1-x) - h
\end{aligned}$$

Why are x, τ and h dimensionless ? (Hint: Consider how they are defined, and the units of N, t and H).

(b) The fixed points are

$$x_e = \frac{1}{2} \pm \sqrt{\frac{1}{4} - h}$$

with multipliers

$$f'(x_e) = 1 - 2x_e = \mp\sqrt{\frac{1}{4} - h}$$

respectively. Hence there is a saddle node bifurcation at $h = h_c = 1/4$.

(c) When $h < h_c$ there are 2 fixed points. The smaller (call it x_s)is unstable. In fact for $x(0) < x_s$, we get $x(t) \to -\infty$ as $t \to \infty$, which is what makes the model unrealistic. The larger fixed point is stable.
When $h > h_c$, there are no fixed points, due to overfishing; (again we get the unrealistic $x(t) \to -\infty$ as $t \to \infty$.)

3. (a)

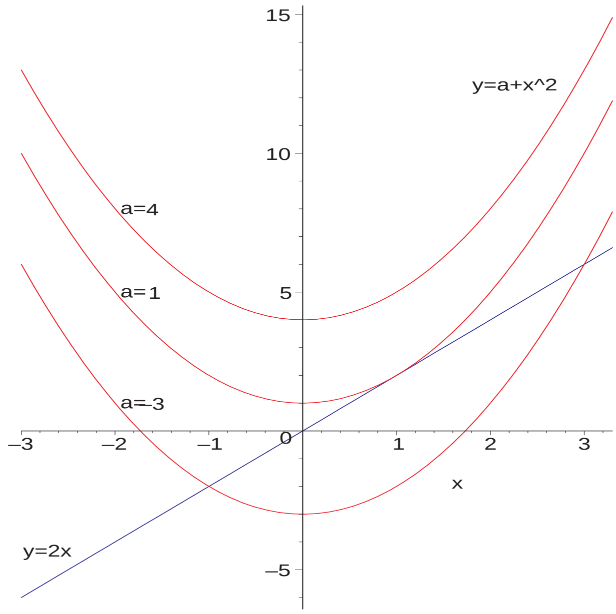

Figure 6: Nullclines for Vector Field of Q3

It looks like there is a saddle node bifurcation at $a = 1$.

(b) $\mathbf{x}_e = (x_e,\ 2x_e)^T$ where

$$0 = a + x_e^2 - 2x_e \quad \Rightarrow \quad x_e = 1 \pm \sqrt{1-a}$$

So, saddle node ("blue sky") bifurcation occurs at $a = a_c = 1$. As a check, the Jacobian of the vector field is

$$\begin{pmatrix} -2 & 1 \\ 2x_e & -1 \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 + 3\lambda + 2(1 - x_e) = \left(\lambda + \frac{3 + \sqrt{1+8x_e}}{2}\right)\left(\lambda + \frac{3 - \sqrt{1+8x_e}}{2}\right)$$

At $a = a_c,\ x_e = 1$ and the eigenvalues are $0,\ -3$.
At $a = 1 - \epsilon^2 < a_c, x_e = 1 \pm \epsilon$ and the eigenvalues are approximately $0 \pm 2\epsilon/3,\ -3 \pm 2\epsilon/3$ [1]. Thus one fixed point is a saddle, while the other is a stable node.

4. $\mathbf{x}_e = (x_e,\ ax_e)^T$ where

$$0 = -abx_e(1 + x_e) + x_e \quad \Rightarrow \quad x_e = 0,\ \frac{1}{ab} - 1$$

The Jacobian of the vector field is

$$\begin{pmatrix} -a & 1 \\ 1/(1+x_e)^2 & -b \end{pmatrix} \quad \Rightarrow \quad \chi(\lambda) = \lambda^2 + (a+b)\lambda + ab - \frac{1}{(1+x_e)^2}$$

At a zero-eigenvalue bifurcation,

$$\begin{aligned} ab - \frac{1}{(1+x_e)^2} &= 0 \\ \Rightarrow ab &= \begin{cases} 1, & \text{if } x_e = 0 \\ 0 \text{ or } 1, & \text{if } x_e = 1/ab - 1 \end{cases} \end{aligned}$$

[1] where the approximation $\sqrt{1 \pm z} \approx 1 \pm z/2$ for small z has been used

Since in the last case ab cannot be 0, we get that

$$(ab)_c = 1, \qquad x^* = 0, \text{ and } \qquad \lambda_c = 0, \; -(a + \frac{1}{a})$$

Let's look at what happens to each of the fixed points as ab crosses through the critical value 1. The eigenvalues of the Jacobian are given by

$$\lambda = -\frac{a+b}{2} \pm \sqrt{\left(\frac{a-b}{2}\right)^2 + \frac{1}{(1+x_e)^2}}$$

(i) $x_e = 0$: For $ab = 1 - \delta < (ab)_c$, the expression for the eigenvalues becomes

$$\lambda = -\frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 + \delta}$$

which always is a saddle. For $ab = 1 + \delta > (ab)_c$, the expression for the eigenvalues becomes

$$\lambda = -\frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - \delta}$$

which is a stable (respectively unstable) node if $a > 0$ (respectively $a < 0$).

(ii) $x_e = 1/ab - 1$: For $ab = 1 - \delta < (ab)_c$, the expression for the eigenvalues becomes

$$\lambda = -\frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - \delta(1-\delta)}$$

which is a stable (respectively unstable) node if $a > 0$ (respectively $a < 0$). For $ab = 1 + \delta > (ab)_c$, the expression for the eigenvalues becomes

$$\lambda = -\frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 + \delta(1-\delta)}$$

which always is a saddle.

5. The Jacobian at $\mathbf{x}_e = \mathbf{0}$ is

$$\begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \Rightarrow \chi(\lambda) = \lambda^2 - 2a\lambda + a^2 + 1$$

$$\Rightarrow \lambda = a \pm i$$

At $a = a_c = 0$, $\lambda = \pm i$; which hints at a *Hopf* bifurcation. Indeed, the *Andronov - Hopf* theorem confirms this since (i) $\Re(\lambda(a_c)) = 0$. $\Im(\lambda(a_c)) \neq 0$ while (ii) $\frac{d\Re(\lambda)}{da}(a_c) \neq 0$. Furthermore, the $\mathbf{x}_e$ change from a stable focus for $a < a_c$ to an unstable focus at $a > a_c$, which is the primary sign of a supercritical bifurcation.

6. (a) The predator-prey model has 4 fixed points, three of which lie on one or other axis; the fourth is of the form $\mathbf{x}_e = (x_e, \; y_e = \frac{1}{a}\frac{x_e}{1+x_e})$, where x_e is a solution of

$$a(b - x_e)(1 + x_e) = \frac{x_e}{1 + x_e}$$

As illustrated in the figure below, there is a $x_e > 0$ near $x = b$ (and hence $y_e > 0$ also.)

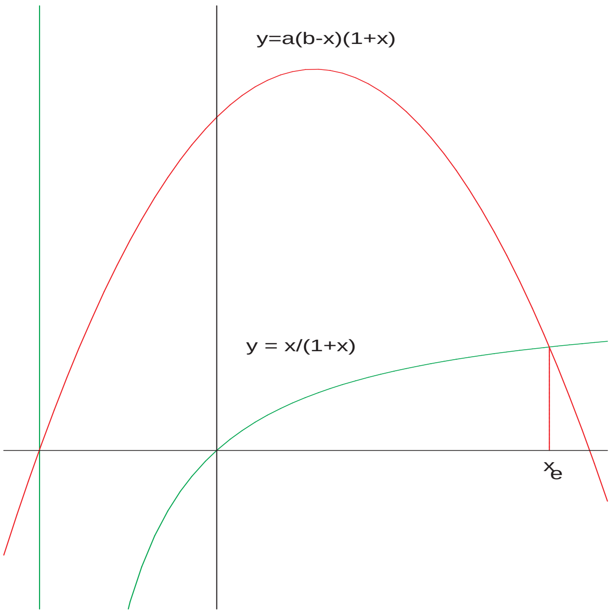

Figure 7: x_e value for Predator-Prey problem

(b) The Jacobian of the vector field at $\mathbf{x}_e$ is

$$\begin{pmatrix} -x_e + x_e y_e/(1+x_e)^2 & -x_e/(1+x_e) \\ y_e/(1+x_e)^2 & -a y_e \end{pmatrix}$$

$$\Rightarrow \quad \chi(\lambda) = \lambda^2 - \frac{x_e}{1+x_e}(b-2-2x_e)\lambda + \frac{x_e}{1+x_e}(2x_e^2 - bx_e + b)$$

At a *Hopf* bifurcation, the coefficient of λ of the characteristic polynomial must be zero. Therefore using the notation $x^* = x_e$ at $a = a_c$ we have $2x^* = b-2 > 0$ and

$$a_c(b-x^*)(1+x^*) = \frac{x^*}{1+x^*} \qquad \Rightarrow \qquad a_c = \frac{4(b-2)}{b^2(b+2)}$$

(c) For $a = a_c - \epsilon$, from the expression for the characteristic polynomial we see that the real part of the eigenvalues satisfies

$$\frac{x_e}{2(1+x_e)}(b-2-2x_e) < \frac{x^*}{2(1+x^*)}(b-2-2x^*) = 0 \qquad \Rightarrow \qquad \Re(\lambda) < 0$$

(use the figure above to see that $x_e < x^*$ as a is reduced). Thus we have a stable focus. Similarly for $a = a_c + \epsilon$, $\Re(\lambda) > 0$ and the fixed is an unstable focus. Thus we have primary evidence for a supercritical bifurcation.