1. For all 4 flows, the only $\mathbf{x}_e = \mathbf{0}$.

   (a) Linear
   $$\chi(\lambda) = \lambda^2 + 3\lambda - 1 = (\lambda + \frac{3+\sqrt{13}}{2})(\lambda + \frac{3-\sqrt{13}}{2})$$
   Saddle; $I = -1$.

   (b) Linear
   $$\chi(\lambda) = \lambda^2 - 4\lambda + 5 = (\lambda - 2 + i)(\lambda - 2 - i)$$
   Unstable focus; $I = 1$.

   (c) Take a small closed curve surrounding $\mathbf{0}$. Traverse it counter clockwise (ccw) starting from the positive x-axis. The vector fields "arrow" goes from 0°ccw to 90°, then cw back to 0°then on to -90°, then finally ccw back to 0°, giving zero net encirclements of $\mathbf{0}$. Thus $I = 0$

   (d) Similar procedure to (c). The line $x + y = 0$ plays an important role. $I = 0$.

2. (a) Vector field has no fixed points; hence no closed orbits.

   (b) Fixed points at

   (i) $\mathbf{0}$. Unstable node. $I = 1$.
   (ii) $(0, 2)^T$. Stable node. $I = 1$.
   (iii) $(4, 0)^T$. Stable node. $I = 1$.
   (iv) $(1, 1)^T$. $\chi(\lambda) = \lambda^2 + 2\lambda - 2 = (\lambda + 1 + \sqrt{3})(\lambda + 1 - \sqrt{3})$. Saddle. $I = -1$

   For this vector field, the x- and y- axes are orbits. Thus no closed orbits can encircle any of the first three fixed points listed above. No closed orbit can encircle a saddle alone. Hence there are no closed orbits.

   (c) Fixed points at

   (i) $\mathbf{0}$. Stable focus. $I = 1$.
   (ii) $(1, 0)^T$. Saddle. $I = -1$.

   The only potential closed orbit is one which encloses $\mathbf{0}$ alone. Use Maple to look at the invariant manifolds of the saddle. No orbit can cut these. (See next question).

3. (a)
$$\begin{aligned} \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x - y + x^2 + y^2) &= -1 + 2y \\ &> 0 \quad \text{if } y > \frac{1}{2} \\ \text{or} \quad &< 0 \quad \text{if } y < \frac{1}{2} \end{aligned}$$

   Hence there are no closed orbits lying (i) entirely above the line $y = 1/2$ or (ii) entirely below the line $y = 1/2$.

(b)

$$\begin{aligned}
\frac{\partial}{\partial x}(e^{\alpha x}y)+\frac{\partial}{\partial y}(e^{\alpha x}(-x-y+x^2+y^2)) &= \alpha e^{\alpha x}y+e^{\alpha x}(-1+2y) \\
&= e^{\alpha x}(-1+(\alpha+2)y) \\
&< 0 \qquad \text{if } \alpha=-2
\end{aligned}$$

Hence there are no closed orbits.

4. (a)

$$\begin{aligned}
r\dot{r} &= x\dot{x}+y\dot{y} \\
&= x(y+x(1-\frac{1}{4}-x^2-y^2))+y(-x+y(1-x^2-y^2)) \\
&= (x^2+y^2)(1-x^2-y^2)-\frac{1}{4}x^2 \\
&= r^2(1-r^2)-\frac{1}{4}r^2\cos^2(\theta) \\
\Rightarrow \qquad \dot{r} &= r(1-r^2-\frac{1}{4}\cos^2(\theta))
\end{aligned}$$

and

$$\begin{aligned}
\dot{\theta} &= \frac{x\dot{y}-y\dot{x}}{r^2} \\
&= \left(x(-x+y(1-x^2-y^2))-y(y+x(1-\frac{1}{4}-x^2-y^2)\right)/r^2 \\
&= \left(-x^2-y^2+\frac{1}{4}xy\right)/r^2 \\
&= -1+\frac{1}{4}\sin(\theta)\cos(\theta) \\
&= -1+\frac{1}{8}\sin(2\theta)
\end{aligned}$$

(b) Firstly note that $\dot{\theta}<0$, and so that all orbits rotate cw. Secondly,

$$\begin{aligned}
\dot{r} &< 0 \\
\Rightarrow 1-r^2-\frac{1}{4}\cos^2\theta &< 0 \\
\Rightarrow r^2 &> 1-\frac{1}{4}\cos^2\theta
\end{aligned}$$

Furthermore since

$$1-\frac{1}{4}\cos^2\theta \le 1$$

then provided $r>1$, $\dot{r}<0$. Similarly when

$$\begin{aligned}
\dot{r} &> 0 \\
\Rightarrow 1-r^2-\frac{1}{4}\cos^2\theta &> 0 \\
\Rightarrow r^2 &< 1-\frac{1}{4}\cos^2\theta
\end{aligned}$$

Furthermore since

$$1 - \frac{1}{4}\cos^2\theta \geq \frac{3}{4}$$

then provided $r < \sqrt{3}/2$, $\dot{r} > 0$. Thus, for a suitably small ϵ, a choice for the annular trapping region is

$$\frac{\sqrt{3}}{2} - \epsilon < r \leq 1 + \epsilon$$

.

(c) The only fixed point of the system is $\mathbf{x}_e = \mathbf{0}$ (why ?)which is outside the trapping region. The flow equations are continuously differentiable. Thus the *Poincaré-Bendixson* Theorem tells us that there is a closed orbit in the trapping region.

(d)

$$T = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{-1 + \frac{1}{8}\sin(2\theta)} = -\frac{16\pi\sqrt{7}}{21}$$

What's the significance of the minus sign ?

5. The surface of section is the positive x-axis. $r_e = 0.90873$ and $P'(r_e) = .000015379$; so the fixed point of the map and hence the periodic orbit of the flow is stable.

   Food for thought: Given a trapping region containing a single periodic orbit, what can be said about the stability of the orbit?