In this sheet, the term "invariant manifold" refers to either a stable, unstable or centre manifold.

1. $\mathbf{x}_e = \mathbf{0}$ is a saddle since the Jacobian A has

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \qquad \Rightarrow \qquad \chi(\lambda) = \lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3)$$

Use *dsolve* to solve

$$\frac{dy}{dx} = \frac{3x - 2y + y^2}{y}, \qquad y(x_0) = y_0$$

where (x_0, y_0) is chosen to be close to $\mathbf{0}$ but lying along the appropriate eigendirection at $\mathbf{0}$.
For example, for the eigenvalue $\lambda = -3$, the associated eigenvector is $\mathbf{e}_\lambda = (1, -3)^T$. Hence with e.g. $\delta = 0.01$, choosing $x_0 = x_e + \delta \mathbf{e}_\lambda(x) = 0.01, y_0 = y_e + \delta \mathbf{e}_\lambda(y) = -0.03$ will yield the stable manifold (or the branch of the stable manifold lying in the 4th quadrant). This approach is in general not very accurate: any small error in (x_0, y_0) will eventually result in significant errors away from the saddle point. There exist special algorithms for the accurate computing of invariant manifolds.

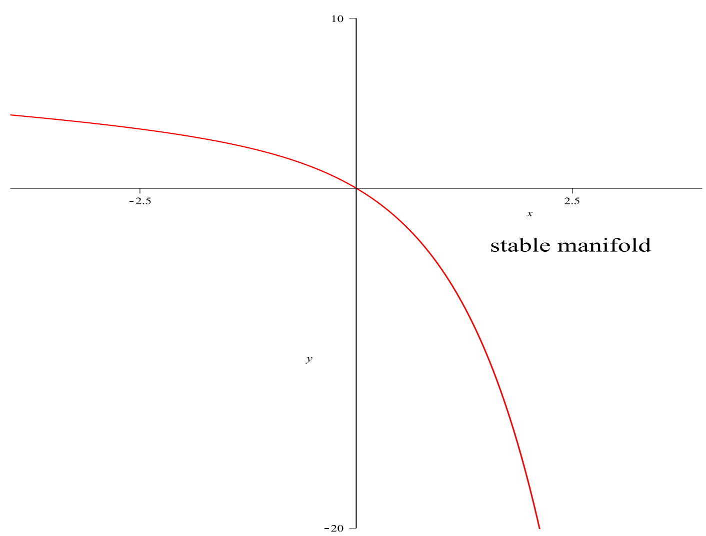

Figure 1: Stable manifold for Q1

2. (a) From the Solution to Q1, the eigenvalues of the Jacobian are $\lambda^S = -3$ and $\lambda^U = 1$. The corresponding eigenvectors are $(1, -3)^T$ and $(1, 1)^T$ respectively. Hence

$$W^S(\mathbf{0}) = \{\alpha(1, -3)^T, \alpha \in \mathbb{R}\}, \qquad W^U(\mathbf{0}) = \{\alpha(1, 1)^T, \alpha \in \mathbb{R}\}.$$

(b) From part (a), the unstable manifold is tangential to the line $y = x$ at $x = 0$. Expressing the manifold as

$$\begin{aligned} y &= h(x) \\ \Rightarrow \qquad \dot{y} &= Dh(x)\dot{x} \\ \Rightarrow \qquad 3x - 2y + y^2 &= Dh(x)y \\ \Rightarrow 3x - 2h(x) + (h(x))^2 &= Dh(x)h(x) \end{aligned} \tag{1}$$

Evaluating Eq (1) at $x = 0$ gives $0 = 0$: no information.
Differentiating Eq (1) with respect to "x" yields

$$3 - 2Dh(x) + 2h(x)Dh(x) = D^2h(x)h(x) + (Dh(x))^2 \tag{2}$$

Evaluating Eq (2) at $x = 0,\ Dh(0) = 1$ (choosing the unstable manifold) gives $3 - 2 \times 1 + 0 = 0 + 1^2$: no new information.
Differentiating Eq (2) with respect to "x" yields

$$-2D^2h(x) + 2(Dh(x))^2 + 2h(x)D^2h(x) = D^3h(x)h(x) + 3D^2h(x)Dh(x) \tag{3}$$

Evaluating Eq (3) at $x = 0,\ Dh(0) = 1$ gives
$-2D^2h(0) + 2(1)^2 + 0 = 0 + 3D^2h(0) \times 1 \Rightarrow D^2h(0) = 2/5$.
Differentiating Eq (3) with respect to "x" yields

$$-2D^3h(x)+6Dh(x)D^2h(x)+2h(D^3h(x) = D^4h(x)h(x)+4D^3h(x)Dh(x)+3(D^2h(x))^2 \tag{4}$$

Evaluating Eq (4) at $x = 0$ gives
$-2D^3h(0) + 6(1)(2/5) + 0 = 0 + 4D^3h(0) \times 1 + 3(2/5)^2 \Rightarrow D^3h(0) = 8/25$.
We have determined that

$$y = x + \frac{1}{5}x^2 + \frac{4}{75}x^3 + O(x^4)$$

(c) At $\mathbf{x}_e = \mathbf{0}$, the *Jacobian* matrix A can be diagonalised:

$$\begin{array}{ccccc} A & = & E & \Lambda & E^{-1} \\ \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} & = & \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} & \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1/4 & -1/4 \\ 3/4 & 1/4 \end{bmatrix} \end{array}$$

Then the change of variable

$$\begin{array}{cccc} \mathbf{X} & = & E^{-1} & \mathbf{x} \\ \begin{pmatrix} X \\ Y \end{pmatrix} & = & \begin{bmatrix} 1/4 & -1/4 \\ 3/4 & 1/4 \end{bmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

yields the flow

$$\dot{X} = -3X - \frac{1}{4}(Y - 3X)^2, \qquad \dot{Y} = Y + \frac{1}{4}(Y - 3X)^2$$

Expressing the stable manifold as

$$\begin{aligned} Y &= h(X) \\ \Rightarrow \qquad \dot{Y} &= Dh(X)\dot{X} \\ \Rightarrow \qquad Y + \frac{1}{4}(Y - 3X)^2 &= Dh(X)\{-3X - \frac{1}{4}(Y - 3X)^2\} \\ \Rightarrow h(X) + \frac{1}{4}(h(X) - 3X)^2 &= Dh(X)\{-3X - \frac{1}{4}(h(X) - 3X)^2\} \end{aligned} \tag{5}$$

We know that $h(0) = 0$ and $Dh(0) = 0$. Differentiating Eq (5) with respect to "X" yields

$$
\begin{aligned}
Dh(X) + \frac{1}{2}(h(X) - 3X)(Dh(X) - 3) \\
&= D^2h(X)\{-3X - \frac{1}{4}(h(X) - 3X)^2\} \\
&\quad + Dh(X)\{-3 - \frac{1}{2}(h(X) - 3X)(Dh(X) - 3)\} \quad (6)
\end{aligned}
$$

On evaluating Eqs (5) and (6) at $h(0) = 0$ and $Dh(0) = 0$, yields $0 = 0$: no new information. Differentiating Eq (6) with respect to "X" yields

$$
\begin{aligned}
D^2h(X) + \frac{1}{2}(Dh(X) - 3)^2 + \frac{1}{2}(h(X) - 3X)D^2h(X) \\
&= D^3h(X)\{-3X - \frac{1}{4}(h(X) - 3X)^2\} \\
&\quad + 2D^2h(X)\{-3 - \frac{1}{2}(h(X) - 3X)(Dh(X) - 3)\} \\
&\quad + Dh(X)\{-\frac{1}{2}(Dh(X) - 3)^2 - \frac{1}{2}(h(X) - 3X)D^2h(X)\} \quad (7)
\end{aligned}
$$

On evaluating Eqs (7) at $h(0) = 0$ and $Dh(0) = 0$, yields

$$D^2h(0) + 9/2 + 0 = 0 + 2D^2h(0)(-3) + 0 \qquad \Rightarrow \qquad D^2h(0) = -9/14$$

Differentiating Eq (7) with respect to "X" yields

$$
\begin{aligned}
D^3h(X) + \frac{3}{2}(Dh(X) - 3)D^2h(X) + \frac{1}{2}(h(X) - 3X)D^3h(X) \\
&= D^4h(X)\{-3X - \frac{1}{4}(h(X) - 3X)^2\} \\
&\quad + 3D^3h(X)\{-3 - \frac{1}{2}(h(X) - 3X)(Dh(X) - 3)\} \\
&\quad + 3D^2h(X)\{-\frac{1}{2}(Dh(X) - 3)^2 - \frac{1}{2}(h(X) - 3X)D^2h(X)\} \\
&\quad + Dh(X)\{-\frac{3}{2}(Dh(X) - 3)D^2h(X) - \frac{1}{2}(h(X) - 3X)D^3h(X)\} \quad (8)
\end{aligned}
$$

On evaluating Eqs (8) at $h(0) = 0$ and $Dh(0) = 0$, yields

$$D^3h(0) + \frac{3}{2}(-3)(-\frac{9}{14}) + 0 = 0 + 3D^3h(0)(-3) + 3(-\frac{9}{14})(-\frac{9}{2}) = 0 \qquad \Rightarrow D^3h(0) = \frac{81}{140}$$

We have determined that

$$Y = -\frac{9}{28}X^2 + \frac{27}{280}X^3 + O(X^4)$$

or in terms of the original coordinates

$$\frac{3x - y}{4} = -\frac{9}{28}\left(\frac{x - y}{4}\right)^2 + \frac{27}{280}\left(\frac{x - y}{4}\right)^3 + O\left(\left(\frac{x - y}{4}\right)^4\right)$$

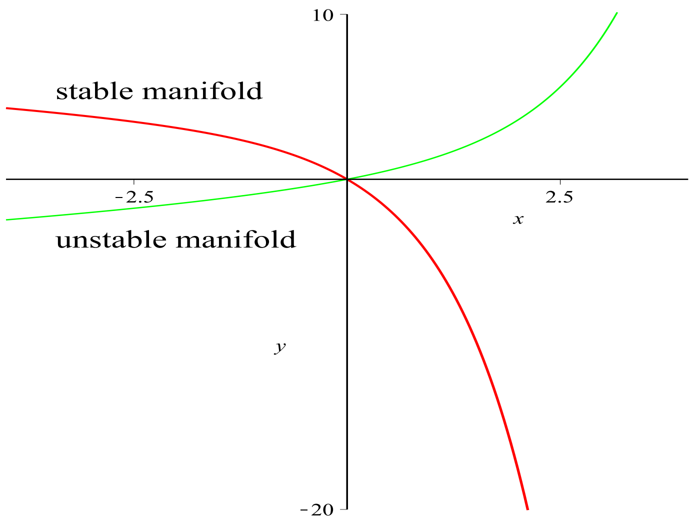

Figure 2: Stable & Unstable manifolds for Q1 & 2

3. For the map $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, let $\mathbf{\Phi}(n, \mathbf{x}_0)$ represent the n-iterate of the orbit starting from $\mathbf{x}_0$. Then if $\mathbf{x}_e$ is a saddle point, the stable manifold $W^s(\mathbf{x}_e)$ is defined by

$$W^s(\mathbf{x}_e) = \{\mathbf{x}; \mathbf{\Phi}(n, \mathbf{x}) \to \mathbf{\Phi}(n, \mathbf{x}_e) = \mathbf{x}_e \ \text{ as } \ n \to \infty\}$$

Similarly, the unstable manifold $W^u(\mathbf{x}_e)$ is defined by

$$W^u(\mathbf{x}_e) = \{\mathbf{x}; \mathbf{\Phi}(n, \mathbf{x}) \to \mathbf{\Phi}(n, \mathbf{x}_e) \ \text{ as } \ n \to -\infty\}$$

Each point on a manifold is on an orbit. Different points may be on different orbits.

4. (a) $\mathbf{x}_e = \mathbf{0}$ is a saddle since the Jacobian A has

$$A = \begin{bmatrix} 0 & 1 \\ 3/4 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \chi(\lambda) = \lambda^2 - \lambda - 3/4 = (\lambda + 1/2)(\lambda - 3/2)$$

(b)

$$\begin{aligned} y &= h(x) \\ \Rightarrow \qquad y' &= h(x') \\ \Rightarrow \qquad \frac{3}{4}x + \frac{1}{4}x^2 - \frac{1}{6}y^2 + y &= h(y) \\ \Rightarrow \frac{3}{4}x + \frac{1}{4}x^2 - \frac{1}{6}\left(h(x)\right)^2 + h(x) &= h(h(x)) \qquad (9) \end{aligned}$$

(c) Substitute $h(x) = -\frac{1}{2}x - \frac{1}{6}x^2$ into Equation(9) to get

$$\frac{3}{4}x + \frac{1}{4}x^2 - \frac{1}{6}(-\frac{1}{2}x - \frac{1}{6}x^2)^2 + (-\frac{1}{2}x - \frac{1}{6}x^2) = -\frac{1}{2}(-\frac{1}{2}x - \frac{1}{6}x^2) - \frac{1}{6}(-\frac{1}{2}x - \frac{1}{6}x^2)^2$$

Simplify to get an identity. Why is the manifold only defined on $-9 < x < 6$?

(d) Any invariant set satisfies the functional equation. The stable manifold is tangential to the linearised stable manifold at the origin. From part (c) $Dh(0) = -1/2$ which is the direction associated with the eigenvector corresponding to the stable eigenvalue $\lambda = -1/2$.

5. As indicated in Q4, the unstable manifold $y = h(x)$ satisfies the functional equation

$$\frac{3}{4}x + \frac{1}{4}x^2 - \frac{1}{6}(h(x))^2 + h(x) = h(h(x)) \tag{10}$$

We know that $h(0) = 0$ and $Dh(0) = 3/2$ (this latter comes from the direction of the eigenvector corresponding to the unstable eigenvector). Differentiating Eq (10) gives

$$\frac{3}{4} + \frac{1}{2}x - \frac{1}{3}h(x)Dh(x) + Dh(x) = Dh(h(x))Dh(x) \tag{11}$$

Evaluating Eqs (10) and (11) at $x = 0$ and $Dh(0) = 3/2$ gives no new information. Differentiating Eq (11) yields

$$\frac{1}{2} - \frac{1}{3}(Dh(x))^2 - \frac{1}{3}h(x)D^2h(x) + D^2h(x) = D^2h(h(x))(Dh(x))^2 + Dh(h(x))D^2h(x) \tag{12}$$

Evaluating Eq (12) at $x = 0$ gives

$$1/2 - (1/3)(3/2)^2 - 0 + D^2h(0) = D^2h(0)(3/2)^2 + (3/2)D^2h(0) \qquad \Rightarrow D^2h(0) = -1/11$$

Differentiating Eq (12) yields

$$\begin{aligned} &- Dh(x)D^2h(x) - \frac{1}{3}h(x)D^3h(x) + D^3h(x) \\ &\quad = D^3h(h(x))(Dh(x))^3 + 3D^2h(h(x))Dh(x)D^2h(x) + Dh(h(x))D^3h(x) \end{aligned} \tag{13}$$

Evaluating Eq (13) at $x = 0$ gives

$$(-3/2)(-1/11) - 0 + D^3h(0) = D^3h(0)(3/2)3 + 3(-1/11)^2(3/2) + (3/2)D^3h(0)$$
$$\Rightarrow D^3h(0) = 96/3751$$

We have determined that

$$y = \frac{3}{2}x - \frac{1}{22}x^2 + \frac{16}{3751}x^3 + O(x^4)$$

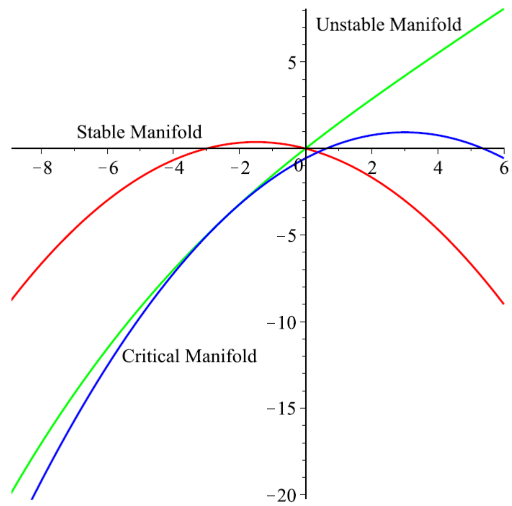

Figure 3: Stable, Unstable & Critical manifolds for Q4 & 5

6. At the fixed point $x_e = 2/3$, we compute $Df(x_e) = -1$; hence x_e is non-hyperbolic. With the change of variable, the new representation of the map is

$$z' = -z - 3z^2 \tag{14}$$

Thus for small z near $z_e = 0$

$$z > 0 \Rightarrow |z'| > |z|$$

$$z < 0 \Rightarrow |z'| < |z|$$

So we cannot tell whether it's stable or not. Consider the second iterate

$$\begin{aligned} z'' &= -z' - 3(z')^2 \\ &= z - 18z^3 - 27z^4 \end{aligned}$$

z^3 always has the same sign as z; thus for small z, $|z''| < |z|$. This argument shows that the subsequence consisting of every second term converges to 0, but the odd numbered terms are related to this convergent subsequence by Eq (14). Hence they converge also and so $z_e = 0$ is asymptotically stable.

7. The *Jacobian* matrix of the map is

$$J = \begin{pmatrix} 3\,[1 - 2x - (1/2)y] & -(3/2)x \\ -(63/16)y & 3.5\,[1 - 2y - (9/8)x] \end{pmatrix}$$

The fixed points of this model with their associated *Jacobian* matrices are readily computed to be

$$\begin{aligned}
\mathbf{x}_1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & J_1 &= \begin{pmatrix} 3 & 0 \\ 0 & 3.5 \end{pmatrix} \\
\mathbf{x}_2 &= \begin{pmatrix} 2/3 \\ 0 \end{pmatrix} & J_2 &= \begin{pmatrix} -1 & -1 \\ 0 & 7/8 \end{pmatrix} \\
\mathbf{x}_3 &= \begin{pmatrix} 0 \\ 5/7 \end{pmatrix} & J_3 &= \begin{pmatrix} 27/14 & 0 \\ -45/16 & -3/2 \end{pmatrix} \\
\mathbf{x}_4 &= \begin{pmatrix} 104/147 \\ -4/49 \end{pmatrix} & J_4 &= \begin{pmatrix} -67/49 & -52/49 \\ -9/28 & 1/7 \end{pmatrix}
\end{aligned} \tag{15}$$

From which we see that $\mathbf{x}_2$ is non-hyperbolic. (what about $\mathbf{x}_4$?)

- Transfer $\mathbf{x}_e$ to $\mathbf{0}$. Let $z = x - 2/3, w = y - 0$ which gives the map

$$z' = -z - w - 3z^2 - \frac{3}{2}zw, \qquad w' = \frac{7}{8}w - \frac{63}{16}zw - \frac{7}{2}w^2$$

- Find a change of variable which decouples the linear component of the map (i.e. diagonalise $A = J_2$):

$$\begin{array}{cccc} A & = & E & \Lambda & E^{-1} \\ \begin{pmatrix} -1 & -1 \\ 0 & 7/8 \end{pmatrix} & = & \begin{pmatrix} 1 & 8 \\ 0 & -15 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 7/8 \end{pmatrix} & \begin{pmatrix} 1 & 8/15 \\ 0 & -1/15 \end{pmatrix} \end{array}$$

  Then the change of variable

$$\begin{array}{ccc} \mathbf{X} & = & E^{-1} & \mathbf{z} \\ \begin{pmatrix} X \\ Y \end{pmatrix} & = & \begin{pmatrix} 1 & 8/15 \\ 0 & -1/15 \end{pmatrix} & \begin{pmatrix} z \\ w \end{pmatrix} \end{array}$$

  yields the map

$$X' = -X - 3(X^2 + 19XY + 60Y^2), \qquad Y' = \frac{7}{8}Y - \frac{63}{16}XY + 21Y^2 \tag{16}$$

- Find the centre manifold W^c; Let $Y = h(X)$ and find h. Note that $Y \equiv 0$ satisfies the second equation of Expression (16). Why does this imply that $Y \equiv 0$ (actually restricted to $0 < x < 1$ -why ?) is the required invariant manifold?

- To determine the stability of the centre manifold: with $Y \equiv 0$, the first equation of Expression (16) becomes

$$X' = -X - 3X^2$$

  We have already analysed this! (see Question 6). Thus we conclude that the fixed point $(X_e = 0, Y_e = 0) = (x_e = 2/3, y_e = 0)$ is asymptotically stable on the centre manifold and, since the other $\lambda = 7/8$, asymptotically stable in a neighbourhood of the fixed point.