UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2016/17

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the interaction between a prey population and a predator population of sizes $x(t),\ y(t) \geq 0$ respectively in some normalised units.

$$\dot{x} = 5x\left(1 - \frac{x}{100}\right) - y\left(\frac{3x}{x + 1.75y}\right), \qquad \dot{y} = y\left(-1 + 0.8\left(\frac{3x}{x + 1.75y}\right)\right).$$

   (a) Find the fixed points of the system. 2

   (b) Use linearisation to classify the non-trivial fixed points. 8

   (c) Show that $y = 0$ and $x = 0$ are trajectories of the system. 2

   (d) Can the system exhibit closed orbits? Explain your answer. 4

2. Consider the production strategy used by a monopolist described by the map

$$x' = x + \alpha(1 - x^2)$$

where $x, x' \geq 0$ are the normalised quantities produced in the current and next periods respectively, $1 - x^2$ is proportional to the marginal profit rate and $0 < \alpha$ is a multiplier (parameter).

   (a) Show that the maximal value that α can have is given by $\alpha = m$, where m satisfies

$$m + \frac{1}{4m} = \frac{1 + \sqrt{1 + 4m^2}}{2m}$$

   and that this is achieved at $m = (3 + 2\sqrt{3})/2 \approx 1.271$. What is the domain of the map at this value? 4

   (b) Find the fixed point of the map, and determine its stability properties as a function of α. 4

   (c) Show that a period-doubling bifurcation occurs at $\alpha = 1$. Show that the resultant 2-cycle is described by

$$x_{1,2} = \frac{1 \pm \sqrt{\alpha^2 - 1}}{\alpha}.$$

   For what range of α values does the 2-cycle exist in the interval of definition? For what range of α values is the 2-cycle stable? And for what value of α is it super stable? 8

3. (a) For the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   where the origin is a saddle point, define its stable and unstable manifolds. 2

(b) By determining the eigenstructure of the matrix appearing below, compute the stable and unstable manifolds of the linear system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \mathbf{x}.$$

6

(c) The system

$$\dot{x} = y, \qquad \dot{y} = 6x - y + xy$$

has a saddle point at the origin whose stable manifold may be described by $y = h(x)$ in a neighbourhood of the origin. Show how a power series approximation to $h(x)$ may be computed, and compute the first two non-zero terms of the series. 8

4. (a) Describe *Lyapunov*'s 2nd or Direct Method for flows, clearly stating the relevant theorems. 2

(b) Show that the origin of

$$\dot{x} = -2y - x^3, \qquad \dot{y} = 2x - y^3$$

is globally asymptotically stable by constructing a quadratic *Lyapunov* function. What does linearisation say about the origin ? 6

(c) State *La Salle*'s Invariance Principle for a flow and discuss how it may be used to determine asymptotic stability of a fixed point of a *Liénard* -type system . 4

(d) Hence or otherwise prove that the origin of

$$\dot{x} = y, \qquad \dot{y} = -x^3 - y\cosh x$$

is asymptotically stable. Is it globally asymptotically stable? 4

5. (a) Show that Index Theory does not preclude the existence of a closed orbit for the system

$$\dot{x} = 2x + y - x^3, \qquad \dot{y} = -x + 2y - y^3$$

4

(b) State the *Poincaré-Bendixson* theorem. 2

(c) By converting to polar coordinates or otherwise, use the theorem to show that

$$\dot{x} = 2x + y - x^3, \qquad \dot{y} = -x + 2y - y^3$$

has a closed orbit by constructing a trapping annulus. 10

6. (a) State the *Andronov-Hopf* bifurcation theorem. 2

(b) A simplified version of the chlorine dioxide-iodine-malonic acid ($Cl\ O_2$-I_2-MA) reaction is given by

$$\dot{x} = a - x - \frac{4xy}{1+x^2}, \qquad \dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

where $x(t)$ and $y(t)$ are the concentrations of the I^- and $Cl\ O_2^-$ ions at time t and a and b are constant parameters which model slowly changing reactants.
For $a = 10$, find the fixed point of the system and classify it as a function of b. 6

(c) Use the theorem to show that the fixed point undergoes a *Hopf* bifurcation at a certain value of the parameter $b = b_c$. Find b_c. 6

(d) Define what is (i) a supercritical and (ii) a subcritical *Hopf* bifurcation. If you are told that the limit cycle born at the bifurcation is stable, is the *Hopf* bifurcation of part(c) super- or subcritical? Explain you answer. 2

7. A discrete-time host-parasite model is given by

$$N_{k+1} = N_k e^{r(1-N_k-P_k)}, \qquad P_{k+1} = \beta N_k P_k$$

where N_k and P_k are the host and parasite densities at time k respectively; r and β are positive real growth parameters.

(a) Find the fixed points of the map. Are they always ecologically meaningful? 4

(b) Investigate the stability of the fixed points with **zero** parasite population as they depend on the parameters. 6

(c) In the case where $\beta = 3/2$, investigate the stability of any fixed points with **non-zero** parasite population as r varies. What bifurcation occurs as the fixed point loses stability? 6