UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2015/16

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 90 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the interaction between the juvenile and adult populations of a species of normalised sizes $x(t), y(t) \geq 0$ respectively.
$$\dot{x} = y\left(1 - \frac{x}{5}\right) - \frac{4x}{5}, \qquad \dot{y} = x\left(1 - \frac{y}{4}\right) - \frac{3y}{4}$$
   (a) Find the fixed points of the system. 4
   (b) Use linearisation to characterise the nature of the fixed points. 8
   (c) Can the system exhibit closed orbits ? Explain your answer. 6

2. Consider a 1-sector *Solow-Swan* economic growth model described by the map
$$x' = \alpha x + \beta f(x)$$
where $x \geq 0$ is the capital per capita in the current period, $0 < \alpha < 1$, $0 < \beta$ are parameters, and $f(x)$ is the production function in intensive form (output per capita).
For the case $f(x) = \sqrt[3]{x}$,
   (a) find the fixed points of the map, and determine their stability properties as a function of the parameters. 8
   (b) Prove that the map is increasing, and hence show that it has no cycles of period-2. 8
   (c) What can be said about other periodic orbits ? 2

3. (a) For the dynamical system
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$
   where the origin is a saddle point, define the stable and unstable manifolds. 2
   (b) Compute the stable and unstable manifolds of the linear system
$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$
   8
   (c) The system
$$\dot{x} = -x + 4y, \qquad \dot{y} = x - y + x^2$$
   has a saddle point at the origin whose unstable manifold which may be described by $y = h(x)$ in a neighbourhood of the origin. Show how a power series approximation to $h(x)$ may be computed, and compute the first two non-zero terms of the series. 8

4. (a) State *La Salle*'s Invariance Principle for a flow and discuss how it may be used to determine asymptotic stability of a fixed point. 4

(b) Using the candidate function $V(x,y) = \frac{y^2}{2} + \int_0^x c(s)ds$, find conditions on $\beta(x)$ which ensure that the origin of

$$\dot{x} = y, \qquad \dot{y} = -c(x) - \beta(x)y$$

is asymptotically stable if (i) $c(x)$ is continuously differentiable and (ii) $xc(x) > 0,\ x \neq 0$ on an interval containing $x = 0$. 9

(c) Hence or otherwise prove that the origin of

$$\dot{x} = y, \qquad \dot{y} = -x(2-x) - (1+x^2)y$$

is asymptotically stable. Is it globally asymptotically stable? 5

5. (a) A "Maternal Effects" model of single species growth is given by

$$N_{k+1} = 2N_k \frac{q_k}{1+q_k}, \qquad q_{k+1} = 4q_k^{\beta} \frac{1}{1+N_k}$$

with the parameter β satisfying $0 < \beta \leq 1$. N_k and q_k are the population density and average "quality" at generation k respectively. Find the fixed points of the system as functions of β and classify them. 9

(b) For the system of part (a), use the *Neimark-Sacker* theorem to show that the positive fixed point undergoes a *Neimark-Sacker* bifurcation at a certain value of the parameter $\beta = \beta_c$. Find the value β_c. 9

6. (a) Show that Index Theory does not preclude the existence of a closed orbit for the Brusselator, a theoretical model of an autocatalytic reaction exemplified by

$$\dot{x} = 1 + x^2y - 4x, \qquad \dot{y} = 3x - x^2y.$$

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(b) Draw the nullclines for the system of part (a), and plot typical flow directions along these nullclines and along the boundary of the irregular pentagon with vertices $(1/4, 0)$, $(1/4, 12)$, $(3/2, 12)$, $(c, (4c-1)/c^2)$ and $(c, 0)$, where $c \approx 13.2$ is the positive root of $z^3 - (27/2)z^2 + 4z - 1 = 0$ (see Fig.1). 7

(c) Use the *Poincaré-Bendixson* theorem to prove that the system of part (a) has a closed orbit. 7

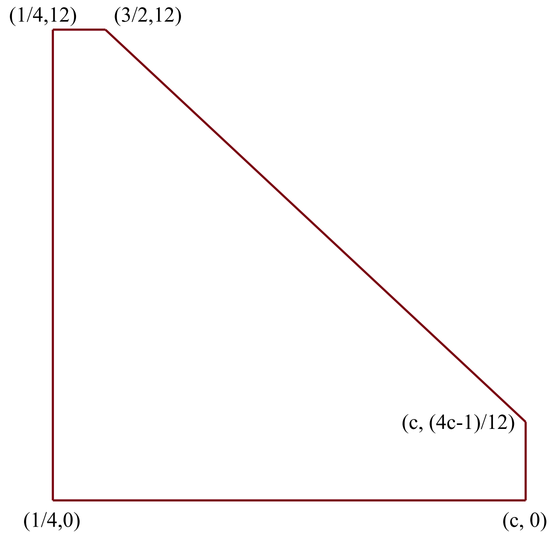

Figure 1: The pentagon of Q 6(b) - (not to scale)

7. The equation

$$\frac{dS}{dt} = rS\left[1 - \left(\frac{S}{K}\right)^{\nu}\right] - CS$$

where $r,\ K,\ \nu$ and $C > 0$ is a model of malign tumour growth which is being treated by chemotherapy, where the dosage is proportional to the tumour size.
$S(t)$ is the tumour size at time t. In the absence of therapy ($C = 0$), the tumour growth is modelled by a generalised logistic curve (r is the growth rate, K is the carrying capacity and ν is a dimensionless fitting parameter). C is the rate of therapeutic dosage per unit time per unit mass of the tumour.

   (a) Show that the system can be rescaled as

   $$\frac{dx}{d\tau} = x(1 - x^{\nu}) - cx$$

   for suitably defined dimensionless quantities $x,\ \tau$ and c. 5

   (b) Find the fixed points as a function of the parameter c, and show that a bifurcation occurs; classify it and determine the critical value $c = c^*$. 9

   (c) Comment on the interpretation of the model when (i) $c < c^*$ and (ii) $c > c^*$ 4