UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2015

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 90 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the mutualistic interaction between two self-limiting species of sizes $x(t),\ y(t) \geq 0$ respectively in normalised units.

$$\dot{x} = x(3 - 2x + y), \qquad \dot{y} = y(2 - y + x).$$

   (a) Find the fixed points of the system. 3

   (b) Use linearisation to characterise the nature of the fixed points. 8

   (c) Show that $y = 0$ and $x = 0$ are trajectories of the system. 2

   (d) Can the system exhibit closed orbits? Explain your answer. 5

2. In traffic models, road occupancy is a measure of the fraction of time that a particular point on a road is occupied by a vehicle. This will change as the number of traffic regimes varies all the way from free flow to jam conditions. Consider the occupancy model, defined on the interval $0 \leq x \leq 1$, by the map

$$x' = \alpha x(1 - x^2)$$

where x is the average occupancy and the parameter $\alpha > 0$ measures average traffic speed relative to free flow speed.

   (a) Show that $\alpha \leq 3\sqrt{3}/2$ to ensure that the orbit of x remains in $[0,\ 1]$ for all initial values in this interval. 2

   (b) Find the fixed points of the map and determine their stability properties as a function of the parameter α. 8

   (c) Show that there is a two-cycle consisting of the points $x_1,\ x_2$ where

$$x_{1,2} = \sqrt{\frac{1 \pm \sqrt{1 - (2/\alpha)^2}}{2}}$$

   whenever $\alpha > 2$.
   Further show that this cycle is stable for $2 < \alpha < \sqrt{5}$. 8

3. (a) For the invertible discrete dynamical system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   where the origin is a saddle point, define the stable and unstable manifolds. 2

   (b) Compute the stable and unstable manifolds of the linear system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ \frac{5}{4} & 2 \end{pmatrix} \mathbf{x}.$$

   6

   (c) Show that the system

$$x' = y, \qquad y' = \frac{5}{4}x + 2y + \frac{5}{2}x^2 - y^2$$

   has a saddle point at the origin, and find and classify any other fixed points. 4

(d) Show that the origin has a quadratic invariant manifold which may be written as

$$y = h(x) = ax + bx^2, \qquad -3/2 < x < 1.$$

and find the values of a and b. Which manifold is it? 6

4. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the dynamical system 2

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

(b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 2

(c) Determine for what value(s) of β is $V(x,y) = \frac{\beta x^2}{1+x^2} + y^2$ a strict *Lyapunov* function for the origin of the system

$$\dot{x} = (y-1)x^3, \qquad \dot{y} = \frac{-3x^4}{(1+x^2)^2} - \frac{y}{1+y^2}\,.$$

Is the origin globally asymptotically stable? 6

(d) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 2

(e) Using the candidate function $V(x,y) = x^4 + y^4$, show that the origin of

$$\dot{x} = -x - y^3, \qquad \dot{y} = x^3$$

is asymptotically stable. Is it globally asymptotically stable? 6

5. (a) State the *Poincaré-Bendixson* theorem. 2

(b) Use the theorem to show that the system

$$\dot{x} = x(2y + 3 - x^2 - y^2) - y, \qquad \dot{y} = y(2y + 3 - x^2 - y^2) + x$$

has a limit cycle. 8

(c) Using the positive x-axis as a surface of section, without solving any equations, show how the *Poincaré* map associated with the flow of part (b) may be computed, and the stability of its fixed point determined. 8

6. Consider the *Lauwerier* system

$$x' = \alpha x(1-x) - xy, \qquad y' = \alpha xy$$

where $x,\ y \geq 0$ and $\alpha > 0$ is a parameter.

(a) Find the fixed points of the system as functions of α and find for what values of α they all coexist. 2

(b) Classify these coexisting fixed points as functions of α. 7

(c) State the *Neimark-Sacker* bifurcation theorem for a two-dimensional map. 2

(d) Use the theorem to show that the positive fixed point of the *Lauwerier* map undergoes a *Neimark-Sacker* bifurcation at a certain value of the parameter $\alpha = \alpha_c$ and find α_c. 7

7. The *Lorenz* system

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = (\rho - z)x - y, \qquad \dot{z} = xy - bz$$

where $\sigma,\ \rho,\ b > 0$ are parameters is the prototype for chaos in flows. It originally arose from a simplified model of a two dimensional fluid flow with prescribed height and temperature difference. If $\sigma = 10$ and $b = 8/3$,

(a) show that the origin is the only fixed point when $\rho < 1$. Classify it and show that

$$V(x, y, z) = \frac{1}{2\sigma}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$$

is a global strict *Lyapunov* function. 6

(b) Show that the origin is a saddle when $\rho > 1$ and find the other fixed points and verify that their characteristic equation is given by

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + \rho)\lambda + 2\sigma b(\rho - 1) = 0$$

4

(c) If in addition $\rho = 28$, the system is known to be chaotic with the non-trivial fixed points having eigenvalues $\{-13.9, 0.1 \pm 10.2i\}$ (correct to one decimal place). If ρ is reduced, show that a bifurcation occurs at a value $\rho = \rho_c$. Find ρ_c and identify the type of bifurcation. 6

(d) Is the system chaotic after the bifurcation? 2