UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018

SEMESTER: Spring 2013

MODULE TITLE: Dynamical Systems

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the interaction between the juvenile and adult populations of a species of sizes $x(t),\ y(t) \geq 0$, respectively, in some normalised units.

$$\dot{x} = 2y\left(1 - \frac{3x}{3+y}\right), \qquad \dot{y} = 2x\left(1 - \frac{y}{3+x}\right).$$

   (a) Find the fixed points of the system. 4

   (b) Use linearisation to characterise the nature of the fixed points. 10

   (c) Can the system exhibit closed orbits ? Explain your answer. 6

2. Consider the following specific instance of the *Gabisch* multiplier-accelerator macroeconomic model

$$x' = x\left(\frac{a - 0.5\sqrt{x}}{a-1}\right),$$

   where x is the national product at year k and $a > 1$ is the "accelerator" parameter.

   (a) Find the fixed points of the map, and determine their stability properties as a function of the parameter a. 8

   (b) Classify any bifurcations that occur for these fixed points, and sketch their bifurcation diagrams over the relevant a-interval. 8

   (c) State *Sharkovsky*'s theorem. It can be shown that there are no period-4 points for $a > A = 1.21$.What can be said about other periodic orbits for $a > A$? 4

3. (a) For the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   where the origin is a saddle point, define the stable and unstable manifolds. 4

   (b) Compute the stable and unstable manifolds of the linear system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \mathbf{x}.$$

   8

(c) The system

$$\dot{x} = 2y(1-3x), \qquad \dot{y} = 2x(1-y)$$

with a saddle point at the origin, has an unstable manifold which may be written as $y = h(x)$ in a neighbourhood of the origin. Show how a power series approximation to the function h may be computed, and compute the first two non-zero terms of this power series. 8

4. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

2

(b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 4

(c) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 6

(d) Determine whether the origin of

$$\dot{x} = y, \qquad \dot{y} = -x|x| - \sin y - y$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 8

5. (a) Use Index Theory to show that the nonlinear system given by

$$\dot{x} = 7 - x - \frac{4xy}{1+x^2}, \qquad \dot{y} = \frac{x}{2}\left(1 - \frac{y}{1+x^2}\right)$$

may have a closed orbit. 4

(b) Draw the nullclines for the system of part (a), and plot typical flow directions along these nullclines and along the boundary of the rectangle $[0,7] \times [0,50]$. 6

(c) State the *Poincaré-Bendixson* theorem, 4
and use it to prove that the system of part (a) has a closed orbit. 6

6. (a) State the *Neimark-Sacker* bifurcation theorem for a two-dimensional map. 4

(b) A "Maternal Effects" model of single species growth is given by

$$N_{k+1} = 2N_k \frac{q_k}{1+q_k}, \qquad q_{k+1} = 3q_k^{\beta} \frac{1}{1+N_k}$$

with the parameter β satisfying $0 < \beta \leq 1$. N_k and q_k are the population density and average "quality" at generation k respectively. Find the fixed points of the system as functions of β and classify them. 8

(c) For the system of part (b), use the *Neimark-Sacker* theorem to show that the positive fixed point undergoes a *Neimark-Sacker* bifurcation at a certain value of the parameter $\beta = \beta_c$. Find the value β_c. 8

7. (a) Describe the *Ott-Grebogi-Yorke* (OGY) control method as it might be applied to the dynamical system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}; p)$$

with parameter p to stabilise the unstable fixed point $\mathbf{x}_e$ (corresponding to the nominal value $p = p_0$) which is embedded in a chaotic attractor. 10

(b) What advantages and disadvantages does the method have? 5

(c) The map $x' = pT(x)$ where T is the tent map

$$T(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1/2, \\ 2 - 2x, & \text{if } 1/2 < x \leq 1 \end{cases}$$

is known to be chaotic for $p > 1/2$. If the system is being driven at the nominal parameter value $p_0 = 1$, construct the OGY controller that stabilises the non trivial fixed point x_e making it superstable when "active": $|x - x_e| < \epsilon$. Find the value for ϵ that corresponds to an allowable parameter variation of $\pm 1\%$. 5