UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4018 | SEMESTER: Spring 2010

MODULE TITLE: Dynamical Systems | DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke | PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Dr. P. Howell

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**

Unless otherwise stated, the following notation is used throughout: $\dot{\mathbf{x}}$ is the time derivative of the continuous-time vector variable $\mathbf{x}(t)$, while $\mathbf{x}'$ is the next value in a discrete-time vector sequence whose current value is $\mathbf{x}$; similarly for scalar variables.

1. The following dynamical system represents the interaction between the juvenile and adult populations of a species of sizes $x(t),\ y(t) \geq 0$ respectively in some normalised units.

$$\dot{x} = y\,(1 - 2x) - \frac{4}{5}x, \qquad \dot{y} = x\left(\frac{2}{3} - 3y\right) - \frac{1}{2}y$$

   (a) Find the fixed points of the system. 6

   (b) Use linearisation to characterise the nature of the fixed points. 8

   (c) Can the system exhibit closed orbits? Explain your answer. 6

2. In traffic models, road occupancy is a measure of the fraction of time that a particular point on a road is occupied by a vehicle. This will change as the number of traffic regimes varies all the way from free flow to jam conditions. Consider the 2-regime occupancy model, defined on the interval $0 \leq x \leq 1$, by the map

$$x' = \begin{cases} \lambda x(1-x), & \text{if } 0 \leq x \leq x_c, \\ \lambda x_c(1-x), & \text{if } x_c < x \leq 1 \end{cases}$$

   where x is the average occupancy and the parameters $\lambda > 0$ and $0 < x_c < 1$ measure average traffic speed relative to free flow speed and the crossover occupancy between regimes respectively. Assume that $x_c = 0.6$.

   (a) Find the fixed points of the map and plot a bifurcation diagram for each fixed point as a function of the parameter λ. 10

   (b) Show that there is a two-cycle consisting of the points $x_1,\ x_2$ where

$$x_1 = \frac{1 + \frac{1}{0.6\lambda^2} - \sqrt{(1 + \frac{1}{0.6\lambda^2})^2 - \frac{4}{\lambda}}}{2}, \qquad x_2 = 1 - \frac{x_1}{0.6\lambda}$$

   Indicate how to determine the range of values of λ for which the 2-cycle exists and is stable. 10

3. (a) Define stability and asymptotic stability in the sense of *Lyapunov* for the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

   2

   (b) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems. 4

(c) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 4

(d) Determine whether the origin of

$$\dot{x} = y, \qquad \dot{y} = -2x - x^3 - y|y|$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 10

4. (a) Define a non-hyperbolic fixed point of a map. 2

(b) Show that the map

$$x' = x + \alpha x^3$$

has a non-hyperbolic fixed point at $x = 0$, and determine its stability properties in terms of the parameter α. 6

(c) Show that the system

$$x' = x + \frac{1}{4}y(y^2 + 3x^2), \qquad y' = y + \frac{1}{4}x(x^2 + 3y^2)$$

has a non-hyperbolic fixed point at the origin. By using the transformation

$$z = \frac{1}{2}(x + y), \qquad w = \frac{1}{2}(x - y)$$

rewrite the system equations in terms of z and w, and hence classify the fixed point at the origin. 12

5. (a) State the *Poincaré-Bendixson* theorem. 6

(b) Use the theorem to show that the system

$$\dot{x} = x - y - x^3, \qquad \dot{y} = x + y - y^3$$

has a limit cycle. 14

6. The equation

$$N_{k+1} = \frac{\mu C N_k}{C + (\mu - 1)N_k} - pN_k$$

where μ, C and $p > 0$ is a model of a fishery with *proportional* harvesting. N_k is the population size in season k. In the absence of harvesting ($p = 0$), the population growth is governed by a Beverton-Holt map where μ is the intrinsic growth rate and C is the carrying capacity. p is the fraction of the fish population harvested per season.

(a) Show that the system can be rescaled as

$$x' = \frac{\mu x}{1+x} - px$$

for a suitably defined dimensionless quantity x. 4

(b) With $\mu = \frac{7}{4}$, find the fixed points of the system of part (a) as a function of the parameter p, and show that a bifurcation occurs; classify it and determine the critical value $p = p_c$. 7

Comment on what the model is saying in the cases when (i) $p < p_c$ and (ii) $p > p_c$. 3

(c) With $\mu = 3$, repeat the analysis of part (b). 6

7. (a) State the *Andronov-Hopf* bifurcation theorem. 4

(b) Use the theorem to show that the fixed point at the origin of

$$\dot{x} = 2y + x(a - x^2 - y^2), \qquad \dot{y} = -2x + y(a - x^2 - y^2)$$

undergoes a *Hopf* bifurcation at a value of the parameter $a = a_c$. Find the value a_c. 9

(c) Define what is (i) a supercritical *Hopf* bifurcation and (ii) a subcritical *Hopf* bifurcation. 2

(d) By expressing the system of part (b) in polar coordinates, determine whether the bifurcation is super- or subcritical. 5