MA4006: Exercise Sheet 5

Please attempt these questions before the tutorials in Weeks 10, 11 and 12.

1. Use the method of separation of variables to solve the wave equation $u_{tt} = c^2 u_{xx}$ for vibrations in an organ pipe subject to the boundary conditions

(a) $u(0,t) = 0$, $t \geq 0$ (the end $x = 0$ is closed);

(b) $\partial u(l,t)/\partial x = 0$, $t \geq 0$ (the end $x = l$ is open);

(c) $u(x,0) = 0$, $0 \leq x \leq l$ (the pipe is initially undisturbed);

(d) $\partial u(x,0)/\partial t = v =$ constant, $0 \leq x \leq l$ (the pipe is given an initial uniform blow).

2. The function $u(x,y)$ satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ in the region $0 \leq x \leq l$, $0 \leq y \leq \infty$, and is zero on the boundary except for $y = 0$, $0 < x < l$, where it takes a constant value u_0. Use the method of separation of variables to show that

$$u(x,y) = \frac{4u_0}{\pi} \sum_{\text{odd}\,n}^{\infty} \frac{1}{n} e^{-n\pi y/l} \sin \frac{n\pi x}{l},$$

and deduce that at $x = l/2$

$$u(l/2, y) = \frac{4u_0}{\pi} \tan^{-1}(e^{-\pi y/l}).$$

[Hint: consider the Maclaurin expansion of $\tan^{-1} x$].

3. Solve the heat equation $u_t = c^2 u_{xx}$ by Laplace transforms subject to

(a) $u(0,t) = 1$, $t > 0$;

(b) $u(x,t) \to 0$ as $x \to \infty$;

(c) $u(x,0) = 0$.

4. Solve the heat equation $u_t = c^2 u_{xx}$ by Fourier transforms with initial condition $u(x,0) = f(x)$ and assuming that $f(x)$ decays at $x = \pm\infty$.

5. Solve the wave equation $u_{tt} = c^2 u_{xx}$ by Laplace transforms subject to $u(x,0) = 0$, $u_t(x,0) = 0$, $u(0,t) = t$ and $u \to 0$ as $x \to \infty$.

6. Consider the finite difference approximation

$$f'(x) \cong \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}.$$

Show that this has $O(h^2)$ accuracy.

7. Consider the heat equation $u_t = u_{xx}$ on $0 \le x \le 1$ and $t > 0$. Use the usual explicit finite difference formulation, i.e. equation (5.15) of the notes, to determine an approximate solution at t = 0.04, if $\Delta t = 0.02$, $\Delta x = 0.2$ and

$$u(x,0) = x^2, \qquad u(0,t) = 0, \qquad u(1,t) = 1.$$

8. Let Ω be the unit square $(0,1) \times (0,1)$, with boundary $\Gamma = [0,1] \times [0,1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega$$

with boundary conditions

$$u(x,y) = g(x,y), \quad (x,y) \in \Gamma.$$

Taking a fixed-width mesh h in both the x and y directions, approximate the first derivatives using a forward difference operator. The 2nd derivative can be approximated using the standard $O(h^2)$ operator. Formulate the discretised problem and outline a uniform mesh.

9. Solve Laplace's equation $\nabla^2 u = 0$ in the unit square with

$$u(x,0) = x, \qquad u(x,1) = 1 - x$$

$$u(0,y) = y, \qquad u(1,y) = 1 - y,$$

with $h = 1/3$. [Use the usual explicit finite difference formulation, i.e. equation (5.20) of the notes].

10. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on $0 \le x \le 1$ and $t > 0$, with

$$u(x,0) = x(1-x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(0,t) = 0, \qquad u(1,t) = 0.$$

Formulate the discretised problem using an explicit finite difference method on a uniform mesh.