MA4006: Exercise Sheet 4

Please attempt these questions before the tutorials in Weeks 8 and 9.

1. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

2. Verify the divergence theorem for the vector field $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

3. Verify Stokes' theorem for the vector field $\mathbf{F} = z\mathbf{i} - 3x\mathbf{j} + 2z\mathbf{k}$, where S is the surface $z = 1 - x^2 - y^2$, $z \geq 0$, C is the boundary circle $x^2 + y^2 = 1$. Assume that S is oriented in the positive z-direction.

4. For each of the following equations, state the order and whether it is linear or nonlinear, and homogeneous or inhomogeneous:

(a) $u_{tt} - u_{xx} + x^2 = 0$

(b) $u_t - u_{xx} + u/x = 0$

(c) $u_x(1 + u^2)^{1/2} + u_y(1 + u_y^2)^{-1/2} = 0$

(d) $u_t - u_{xx} + xu = 0$

5. Solve the following psuedo (or degenerate) PDEs assuming $u = u(x, y)$.

(a) $u_x + u = 0$

(b) $u_x - yu = 0$

(c) $u_x + x^2 u = 0$

(d) $u_{xx} + u_x + u = 0$

6. Fully classify the following equations and find and sketch the characteristics (if any) of the following partial differential equations.

(a) $u_t - u_{xx} + 1 = 0$

(b) $u_{tt} + 2u_{xt} - u_{xx} - u_t + u_x = 0$

(c) $u_{xx} + 5u_{yy} = 0$

(d) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$

7. Check whether $u(x,y) = \sinh x \cosh y$ and $u(x,y) = x^3 - 3xy^2 + 6x^2y - 2y^3$ are solutions to the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

8. Consider the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Use the transformation $\alpha = x - 2y$, $\beta = x - y$ to reduce the equation to a simpler form and determine the general solution $u(x,y)$.

9. Classify the PDE

$$u_{xx} + 2ku_{xy} + k^2u_{yy} = 0, \quad k \neq 0.$$

Using a suitable transformation of form $\xi = x + ay$; $\eta = x + by$, show that the equation can be transformed into the form

$$u_{\xi\xi} = 0\ ,$$

and hence find the solution of the equation in terms of two arbitrary functions.

10. Solve $u_t = c^2u_{xx}$ using the method of separation of variables subject to the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$ and the initial condition $u(x,0) = x(2-x)$.