MA4006: Exercise Sheet 2

Please attempt these questions before the tutorials in Week 3.

1. If $f(x,y) = ax^2 + 2hxy + by^2$, where a, b and h are all constants, find all the first and second order partial derivatives and demonstrate that $f_{xy} = f_{yx}$.

2. If $u = (Ar^n + Br^{-n})\cos(n\theta)$, where A, B, n are constants, show that

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

3. Find the directional derivative of

$$f(x,y,z) = 3xy + 2z^2 + 4x,$$

at the point $P(1,0,-2)$ in the direction of the vector $\mathbf{a} = \mathbf{j} - 2\mathbf{k}$.

4. Explain the term *level surfaces* in relation to any scalar field ω. Interpret $\nabla\omega$ with regard to level surfaces. Hence, find the unit normal to the surface defined by $z = \sqrt{2xy + x^2}$, at the point $(2,1,2)$.

5. Using the fact that $\nabla\Omega$ (where $\Omega = \Omega(x,y)$ is a scalar valued function of position) is perpendicular to it's own level curves, find a unit normal to the curve defined by $x + y = 1$ at the point $(1,0)$. Verify this result with a rough sketch.

6. Find the derivative of the scalar field $\Omega = x^2yz + 4xz^2$ in the direction of the vector $(2,-1,-1)$ at the point $(1,-2,-1)$.

7. Find a unit normal to the surface $z = 3x^2y + x$ at the point $(1,1,4)$.

8. Use Taylor's series in two dimensions to find a first order approximation for $f(2.5,1)$ based on quantities evaluated at the point $(2,1.5)$ in the case where

$$f(x,y) = x^2y + e^{xy}.$$

9. Use Taylor's series in two dimensions to find a first order approximation of

$$f(x,y) = \sin(2xy) + e^{x+3y},$$

at the point $(1,0)$. Hence approximate $f(1.5,0.2)$ and estimate the size of the error.

10. If $\Omega = xy^3z^3 + x^3y^2z$, find grad Ω at the point $(1, -1, 1)$.

11. If $\Omega = x^n + y^n + z^n$ (with n a known constant) and if $\mathbf{r} = (x, y, z)$ then show that $\mathbf{r} \cdot \nabla\Omega = n\Omega$.

12. Find the divergence and curl of $\mathbf{f} = (xy, yz, 0)$ at the point $(1, 1, 1)$. Also evaluate $\nabla(\nabla \cdot \mathbf{f})$.

13. If $\Omega = x + y^2 + z^3$, find div(gradΩ) and curl(gradΩ).