MA4006: Exercise Sheet 5

Please attempt these questions before the tutorials in Weeks 10,11 and 12.

1. Use the method of separation of variables to solve the wave equation $u_{tt} = c^2 u_{xx}$ for vibrations in an organ pipe subject to the boundary conditions

(a) $u(0,t) = 0$, $t \geq 0$ (the end $x = 0$ is closed);

(b) $\partial u(l,t)/\partial x = 0$, $t \geq 0$ (the end $x = l$ is open);

(c) $u(x,0) = 0$, $0 \leq x \leq l$ (the pipe is initially undisturbed);

(d) $\partial u(x,0)/\partial t = v =$ constant, $0 \leq x \leq l$ (the pipe is given an initial uniform blow).

Solution. Let $u(x,t) = F(x)G(t)$. *Then* $u_{tt} = F(x)\ddot{G}(t)$ *and* $u_{xx} = F''(x)G(t)$. *Substituting these into* $u_{tt} = c^2 u_{xx}$ *gives*

$$F(x)\ddot{G}(t) = c^2 F''(x)G(t) \qquad \Longrightarrow \qquad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k,$$

or

$$\ddot{G} - c^2 k G = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u(0,t) = u_x(l,t) = 0$ *therefore imply* $F(0) = F'(l) = 0$.

- $k = 0$: *The solution of* $F'' = 0$ *is* $F(x) = Ax + B$. *The bcs* $F(0) = F'(l) = 0$ *imply that* $A = B = 0$ *and so there are no non-trivial solutions.*

- $k = p^2 > 0$: *The solution of* $F'' - p^2 F = 0$ *is* $F(x) = Ae^{px} + Be^{-px}$. *The bc* $F(0) = 0$ *implies* $A + B = 0$ *and the bc* $F'(l) = 0$ *implies* $Ape^{pl} - Bpe^{-pl} = 0$. *Then* $A = B = 0$ *and so there are no non-trivial solutions.*

- $k = -p^2 < 0$: *The solution of* $F'' + p^2 F = 0$ *is* $F(x) = A\cos px + B\sin px$. *The bc* $F(0) = 0$ *implies* $A = 0$ *and the bc* $F'(l) = 0$ *implies* $B\cos(pl) = 0$, *and so* $pl = (n + 1/2)\pi$, *for* $n = 0, 1, 2, \ldots$. *Thus*

$$F_n(x) = B_n \sin\frac{(n+1/2)\pi x}{l}.$$

Now solve the ODE for G. We have

$$\ddot{G} + \frac{c^2(n+1/2)^2\pi^2}{l^2} G_n = 0 \qquad \Longrightarrow \qquad G_n(t) = \bar{C}_n \cos(\lambda_n t) + \bar{D}_n \sin(\lambda_n t),$$

where $\lambda_n = c(n+1/2)\pi/l$. *Hence*

$$u_n(x,t) = F_n(x)G_n(t) = \big[C_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)\big)\big] \sin \frac{(n+1/2)\pi x}{l},$$

where $C_n = \bar{C}_n B_n$ *and* $D_n = \bar{D}_n B_n$. *This gives*

$$u(x,t) = \sum_{n=0}^{\infty} \big[C_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)\big)\big] \sin \frac{(n+1/2)\pi x}{l}.$$

Finally, to find C_n *and* D_n *we use the initial conditions* $u(x,0) = 0$ *and* $u_t(x,0) = v$. *The first implies*

$$0 = \sum_{n=0}^{\infty} C_n \sin \frac{(n+1/2)\pi x}{l},$$

and so $C_n = 0$. *Now*

$$\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} \lambda_n D_n \cos(\lambda_n t) \sin \frac{(n+1/2)\pi x}{l}.$$

Thus

$$v = \sum_{n=1}^{\infty} \lambda_n D_n \sin \frac{(n+1/2)\pi x}{l},$$

and so

$$\lambda_n D_n = \int_0^l v \sin \frac{(n+1/2)\pi x}{l}\, dx = \frac{2v}{(n+1/2)\pi},$$

or

$$D_n = \frac{2lv}{\pi^2 c(n+1/2)^2}.$$

2. The function $u(x,y)$ satisfies the Laplace equation $u_{xx} + u_{yy} = 0$ in the region $0 \le x \le l$, $0 \le y \le \infty$, and is zero on the boundary except for $y = 0$, $0 < x < l$, where it takes a constant value u_0. Use the method of separation of variables to show that

$$u(x,y) = \frac{4u_0}{\pi} \sum_{\text{odd} n}^{\infty} \frac{1}{n} e^{-n\pi y/l} \sin \frac{n\pi x}{l},$$

and deduce that at $x = l/2$

$$u(l/2, y) = \frac{4u_0}{\pi} \tan^{-1}(e^{-\pi y/l}).$$

[Hint: consider the Maclaurin expansion of $\tan^{-1} x$].

Solution. Let $u(x,y) = F(x)G(y)$. Then $u_{yy} = F(x)\ddot{G}(y)$ and $u_{xx} = F''(x)G(y)$. Substituting these into $u_{xx} + u_{yy} = 0$ gives

$$F''(x)G(y) + F(x)\ddot{G}(y) = 0 \qquad \Longrightarrow \qquad -\frac{\ddot{G}}{G} = \frac{F''}{F} = k,$$

or

$$\ddot{G} + kG = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u(0,y) = u(l,y) = 0$ therefore imply $F(0) = F(l) = 0$. So, for F the only nontrivial solutions are for $k = -p^2 < 0$. We therefore find that $p = n\pi/l$ (standard from the lecture notes) and

$$F_n(x) = B_n \sin\frac{n\pi x}{l}\ , \qquad n = 1, 2, 3, \ldots.$$

Thus we must solve $\ddot{G} - p^2 G = 0$ or $\ddot{G} - \lambda_n^2 G = 0$, where $\lambda_n = n\pi/l$. This has solution

$$G_n(y) = C_n e^{\lambda_n y} + D_n e^{-\lambda_n y}.$$

The b.c. $u(x,\infty) = 0$ implies that $G(y) \to 0$ as $y \to \infty$. Thus we need $C_n = 0$ and so

$$u(x,y) = \sum_{n=1}^{\infty} D_n e^{-n\pi y/l} \sin\frac{n\pi x}{l}.$$

The BC $u(x,0) = u_0$ implies that

$$u_0 = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l} \qquad \Longrightarrow \qquad D_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l} dx = \frac{u_0}{n\pi}\big[1 - \cos n\pi\big],$$

for $n = 1, 2, 3, \ldots$, and so

$$u(x,y) = \frac{4u_0}{\pi}\sum_{odd n} e^{-n\pi y/l} \sin\frac{n\pi x}{l}.$$

At $x = l/2$ we have

$$u(l/2,y) = \frac{4u_0}{\pi}\sum_{odd n} e^{-n\pi y/l} \sin\frac{n\pi}{2} = \frac{4u_0}{\pi}\left(e^{-\pi y/l} - \frac{1}{3}e^{-3\pi y/l} + \frac{1}{5}e^{-5\pi y/l} + \ldots\right).$$

The Maclaurin expansion of $\tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,$$

for $|x| < 1$ *and since* $|e^{-\pi y/l}| < 1$ *we have*

$$u(l/2, y) = \frac{4u_0}{\pi}\tan^{-1}(e^{-\pi y/l}).$$

3. Solve the heat equation $u_t = c^2 u_{xx}$ by Laplace transforms subject to

(a) $u(0,t) = 1$, $t > 0$;

(b) $u(x,t) \to 0$ as $x \to \infty$;

(c) $u(x,0) = 0$.

Solution. Let $\hat{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt$. *Then*

$$\begin{aligned}\int_0^\infty u_t(x,t)e^{-st}dt &= \left[u(x,t)e^{-st}\right]_0^\infty + s\int_0^\infty u(x,t)e^{-st}dt \\ &= -u(x,0) + s\int_0^\infty u(x,t)e^{-st}dt \\ &= s\hat{u}(x,s),\end{aligned}$$

since $u(x,0) = 0$. *Hence the PDE becomes*

$$c^2\hat{u}_{xx} - s\hat{u} = 0 \qquad \Longrightarrow \qquad \hat{u}(x,s) = Ae^{-\sqrt{s}x/c} + Be^{\sqrt{s}x/c}.$$

The BC $u(0,t) = 1$ *implies that*

$$\hat{u}(0,s) = \int_0^\infty e^{-st}u(0,t)dt = \int_0^\infty e^{-st}dt = \frac{1}{s}.$$

From the bc $u \to 0$ *as* $x \to \infty$ *it follows that* $\hat{u} \to 0$ *as* $x \to \infty$ *and so* $B = 0$. *Also* $\hat{u}(0,s) = 1/s$ *implies that* $A = 1/s$. *Thus*

$$\hat{u}(x,s) = \frac{1}{s}e^{-\sqrt{s}x/c} \qquad \Longrightarrow \qquad u(x,t) = \mathit{erfc}\left(\frac{x}{2c\sqrt{t}}\right),$$

(from tables).

4. Solve the heat equation $u_t = c^2 u_{xx}$ by Fourier transforms with initial condition $u(x,0) = f(x)$ and assuming that $f(x)$ decays at $x = \pm\infty$.

Solution. Let $\hat{u}(\omega,t) = \int_{-\infty}^\infty u(x,t)e^{-i\omega x}dx$. *Then* $u_t = c^2 u_{xx}$ *becomes*

$$\hat{u}_t = c^2(i\omega)^2\hat{u} = -c^2\omega^2\hat{u},$$

and so

$$\hat{u}(\omega, t) = C(\omega)e^{-c^2\omega^2 t}.$$

The boundary condition $u(x, 0) = f(x)$ implies

$$\hat{u}(\omega, 0) = \int_{-\infty}^{\infty} u(x, 0)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \hat{f}(\omega),$$

i.e. $C(\omega) = \hat{f}(\omega)$. So

$$\hat{u}(\omega, t) = \hat{f}(\omega)e^{-c^2\omega^2 t},$$

and so by the inverse transform

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-c^2\omega^2 t + i\omega x}d\omega.$$

5. Solve the wave equation $u_{tt} = c^2 u_{xx}$ by Laplace transforms subject to $u(x, 0) = 0$, $u_t(x, 0) = 0$, $u(0, t) = t$ and $u \to 0$ as $x \to \infty$.

Solution. Let $\hat{u}(x, s) = \int_0^\infty u(x, t)e^{-st}dt$. Then

$$\begin{aligned}\int_0^\infty u_{tt}(x, t)e^{-st}dt &= \left[u_t(x, t)e^{-st}\right]_0^\infty + s\int_0^\infty u_t(x, t)e^{-st}dt \\ &= -u_t(x, 0) + s\left[u(x, t)e^{-st}\right]_0^\infty + s^2\int_0^\infty u(x, t)e^{-st}dt \\ &= -u_t(x, 0) - su(x, 0) + s^2\hat{u}(x, s) \\ &= s^2\hat{u}(x, s),\end{aligned}$$

since $u(x, 0) = u_t(x, 0) = 0$. Hence the PDE becomes

$$\hat{u}_{xx} - \frac{s^2}{c^2}\hat{u} = 0 \qquad \Longrightarrow \qquad \hat{u} = Ae^{-sx/c} + Be^{sx/c}.$$

From the bc $u \to 0$ as $x \to \infty$ it follows that $B = 0$. The bc $u(0, t) = t$ implies that

$$\hat{u}(0, s) = \int_0^\infty te^{-st}dt = \left[-\frac{1}{s}te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s^2}.$$

Hence $A = 1/s^2$ and so

$$\hat{u} = \frac{1}{s^2}e^{-sx/c} \qquad \Longrightarrow \qquad u(x, t) = H\left(t - \frac{x}{c}\right)\left(t - \frac{x}{c}\right),$$

where $H(\cdot)$ is the Heaviside function (the inverse LT of $1/s^2$ is t).

6. Consider the finite difference approximation

$$f'(x) \cong \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}.$$

Show that this has $O(h^2)$ accuracy.

Solution. Use Taylor Series:

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \ldots$$
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$$

Thus

$$\begin{aligned} f'(x) &\cong \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h} \\ &= \frac{1}{2h}\left\{-\left[f(x)+2hf'(x)+\frac{(2h)^2}{2!}f''(x)+\frac{(2h)^3}{3!}f'''(x)+\ldots\right]\right. \\ &\left.+4\left[f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots\right]-3f(x)\right\} \\ &= \frac{1}{2h}\left\{2hf'(x)-\frac{2}{3}h^3f'''(x)+\ldots\right\} \\ &= f'(x) - O(h^2), \end{aligned}$$

and so the finite difference approximation has $O(h^2)$ accuracy.

7. Consider the heat equation $u_t = u_{xx}$ on $0 \le x \le 1$ and $t > 0$. Use the usual explicit finite difference formulation, i.e. equation (5.15) of the notes, to determine an approximate solution at t = 0.04, if $\Delta t = 0.02$, $\Delta x = 0.2$ and

$$u(x,0) = x^2, \qquad u(0,t) = 0, \qquad u(1,t) = 1.$$

Solution. Let $u_{i,j} = u(x_i, t_j)$, where $x_i = i\Delta x$, $i = 0, 1, \ldots, I$ and $t_j = j\Delta t$, $j = 0, 1, 2, \ldots$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}.$$

Putting these into $u_t = c^2 u_{xx}$ gives

$$\frac{u_{i,j+1} - u_{i,j}}{c^2 \Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j}, \tag{1}$$

where $\lambda = c^2\Delta t/\Delta x^2$ *and* $i = 1, \ldots, I-1$, $j = 0, 1, 2, \ldots$.

The initial condition $u(x, 0) = x^2$ *gives* $u_{i,0} = x_i^2$, $i = 0, 1, \ldots, I$. *The boundary condition* $u(1, t) = 1$ *gives* $u_{I,j} = 1$, *for* $j = 0, 1, 2, \ldots$. *Since* $\Delta x = 0.2$ *we have* $i = 0, 1, \ldots, 5$. *Thus* $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$ *and* $x_5 = 1$. *The boundary conditions* $u(0, t) = 0$, $u(1, t) = 1$ *imply* $u_{0,j} = 0$, $u_{5,j} = 1$, $j = 0, 1, 2, \ldots$. *We need to find* $u_{i,2}$, *which corresponds to the solution at* $t = 0.04$ *(* $u_{i,1}$ *is the solution at* $t = 0.02$, *since* $\Delta t = 0.02$*). Using equation (1) (with* $j = 0$*):*

$$u_{i,1} = \lambda u_{i-1,0} + (1 - 2\lambda)u_{i,0} + \lambda u_{i+1,0},$$

for $i = 1, 2, 3, 4$. *We know that* $u_{0,1} = 0$ *and* $u_{5,1} = 1$ *(from the BCs). Thus*

$$u_{i,1} = \lambda x_{i-1}^2 + (1 - 2\lambda)x_i^2 + \lambda x_{i+1}^2 = \frac{1}{2}(x_{i-1}^2 + x_{i+1}^2),$$

since $\lambda = 1/2$ *for* $i = 1, 2, 3, 4$. *Hence*

$$u_{1,1} = 0.08, \qquad u_{2,1} = 0.2, \qquad u_{3,1} = 0.4, \qquad u_{4,1} = 0.68.$$

Using equation (1) (with $j = 1$*):*

$$u_{i,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1}),$$

for $i = 1, 2, 3, 4$. *We know that* $u_{0,2} = 0$ *and* $u_{5,2} = 1$ *(from the BCs). Thus*

$$u_{1,2} = 0.1, \qquad u_{2,2} = 0.24, \qquad u_{3,2} = 0.44, \qquad u_{4,2} = 0.7.$$

8. Let Ω be the unit square $(0, 1) \times (0, 1)$, with boundary $\Gamma = [0, 1] \times [0, 1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega$$

with boundary conditions

$$u(x, y) = g(x, y), \quad (x, y) \in \Gamma.$$

Taking a fixed-width mesh h in both the x and y directions, approximate the first derivatives using a forward difference operator. The 2nd derivative can be approximated using the standard $O(h^2)$ operator. Formulate the discretised problem and outline a uniform mesh.

Solution. Use central differences for the second derivatives, with $\Delta x = \Delta y = h$. Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}, \qquad \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}.$$

Hence the PDE $u_{xx} + u_{yy} = f(x,y)$ is discretised as

$$u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y) \approx h^2 f(x,y). \qquad (2)$$

The square is discretised by a uniform mesh of width $h = 1/N$. The mesh points are the intersections of the lines (x_i, y_j) where $x_i = ih$ and $y_j = jh$ for some $0 \le i,\, j \le N$. Writing the approximation to $u(x_i, y_j)$ as $u_{i,j}$, and similarly for f, equation (2) becomes

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}, \qquad 1 \le i,\, j \le N-1. \qquad (3)$$

On the boundary we have

$$u_{i,0} = g_{i,0}, \qquad u_{i,N} = g_{i,N}, \qquad u_{0,j} = g_{0,j}, \qquad u_{N,j} = g_{N,j}.$$

9. Solve Laplace's equation $\nabla^2 u = 0$ in the unit square with

$$u(x,0) = x, \qquad u(x,1) = 1 - x$$
$$u(0,y) = y, \qquad u(1,y) = 1 - y,$$

with $h = 1/3$. [Use the usual explicit finite difference formulation, i.e. equation (5.20) of the notes].

Solution. Using $h = 1/3$ means there are just 4 internal points. Since $f = 0$ the difference equation (3) becomes

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = 4u_{i,j}, \qquad 1 \le i,\, j \le 2.$$

The boundary conditions are

$$u_{i,0} = x_i, \qquad u_{i,N} = 1 - x_i, \qquad i = 0,\, 1,\, 2,\, 3$$

$$u_{0,j} = y_j, \qquad u_{N,j} = 1 - y_j, \qquad j = 0,\, 1,\, 2,\, 3.$$

Hence the linear system of equations to solve is

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} = 4u_{1,1}$$

$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} = 4u_{2,1}$$

$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} = 4u_{1,2}$$

$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} = 4u_{2,2}.$$

This determines the unknown interior points $u_{1,1}$*,* $u_{2,1}$*,* $u_{1,2}$ *and* $u_{2,2}$*.*

10. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on $0 \leq x \leq 1$ and $t > 0$, with

$$u(x,0) = x(1-x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(0,t) = 0, \qquad u(1,t) = 0.$$

Formulate the discretised problem using an explicit finite difference method on a uniform mesh.

Solution. Let $u_{i,j} = u(x_i, t_j)$*, where* $x_i = i\Delta x$*,* $i = 0, 1, \ldots, I$ *and* $t_j = j\Delta t$*,* $j = 0, 1, 2, \ldots$*. Then*

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}.$$

Putting these into $u_{tt} = c^2 u_{xx}$ *gives*

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{c^2 \Delta t^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \lambda^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}), \tag{4}$$

where $\lambda = c\Delta t/\Delta x$*,* $\Delta x = 1/I$ *and* $i = 1, \ldots, I-1$*,* $j = 1, 2, \ldots$*. The boundary conditions* $u(0,t) = u(1,t) = 0$ *imply* $u_{0,j} = u_{I,j} = 0$*. The initial condition* $u(x,0) = x(1-x)$ *implies* $u_{i,0} = x_i(1-x_i)$*,* $i = 0, 1, \ldots, I$*. For the initial condition* $u_t(x,0) = 0$ *we use a central difference to deduce that*

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv \frac{u_{i,1} - u_{i,-1}}{2\Delta t} = 0 \qquad \Longrightarrow \qquad u_{i,1} = u_{i,-1}.$$

Then from (4) with $j = 0$ *we find that*

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + \lambda^2 (u_{i+1,0} - 2u_{i,0} + u_{i-1,0}),$$

or

$$u_{i,1} = (1 - \lambda^2) u_{i,0} + \frac{\lambda^2}{2}(u_{i+1,0} - u_{i-1,0}).$$