MA4006: Exercise Sheet 4: Solutions

1. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution. Now

$$\oint_C (xy + y^2)\,dx + x^2\,dy = \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{f} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{f} \cdot d\mathbf{r},$$

where $\mathbf{f} = (xy+y^2, x^2)$ *and* C_1 *and* C_2 *are given in Figure* **??**. *We parameterise* C_1 *as* $x = t$, $y = t^2$, $0 \le t \le 1$. *Then*

$$\mathbf{r}(t) = (t, t^2) \qquad \Longrightarrow \qquad \mathbf{r}'(t) = (1, 2t) \qquad \& \qquad \mathbf{f}(\mathbf{r}(t)) = (t^3 + t^4, t^2).$$

Thus

$$\oint_{C_1} \mathbf{f} \cdot d\mathbf{r} = \int_0^1 \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^1 (t^3 + t^4 + 2t^3)\,dt = \frac{19}{20}.$$

We parameterise C_2 *as* $\mathbf{r}(t) = \mathbf{r}_0 + (\mathbf{r}_1 - \mathbf{r}_0)t$, $0 \le t \le 1$, *where* $\mathbf{r}_0 = (1,1)$ *and* $\mathbf{r}_1 = (0,0)$. *Then*

$$\mathbf{r}(t) = (1-t, 1-t) \qquad \Longrightarrow \qquad \mathbf{r}'(t) = (-1,-1) \qquad \& \qquad \mathbf{f}(\mathbf{r}(t)) = (2(1-t)^2, (1-t)^2).$$

Thus

$$\oint_{C_2} \mathbf{f} \cdot d\mathbf{r} = \int_0^1 -3(1-t)^2\,dt = -1,$$

and so

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \frac{19}{20} - 1 = -\frac{1}{20}.$$

Green's theorem says

$$\oint_C f_1\,dx + f_2\,dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy,$$

where $f_1 = xy + y^2$ *and* $f_2 = x^2$. *So*

$$\frac{\partial f_2}{\partial x} = 2x \ , \qquad \frac{\partial f_1}{\partial y} = x + 2y,$$

and so By Green's theorem

$$\oint_C (xy+y^2)\,dx + x^2\,dy = \iint_R (2x - x - 2y)\,dx\,dy = \int_0^1 \int_{x^2}^x (x - 2y)\,dy\,dx = \int_0^1 (x^4 - x^3)\,dx = -\frac{1}{20}.$$

Figure 1: *Question 1.*

2. Verify the divergence theorem for the vector field $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

Solution. The divergence theorem says $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV$. *Now*

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z.$$

So

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V (4 - 4y + 2z)\, dV.$$

Use cylindrical co-ordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \qquad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad 0 \le z \le 3.$$

So

$$\begin{aligned}
\iiint_V \nabla \cdot \mathbf{F}\, dV &= \int_0^3 \int_0^{2\pi} \int_0^2 (4 - 4r\sin\theta + 2z) r\, dr\, d\theta\, dz \\
&= \int_0^3 \int_0^{2\pi} \left[2r^2 - 2r^2\sin\theta + zr^2\right]_0^2 d\theta\, dz \\
&= \int_0^3 \int_0^{2\pi} \left(8 - 8\sin\theta + 4z\right) d\theta\, dz \\
&= \int_0^3 \left[8\theta + 8\cos\theta + 4z\theta\right]_0^{2\pi} dz \\
&= \int_0^3 \left(16\pi + 8\pi z\right) dz = \left[16\pi z + 4\pi z^3\right]_0^3 = 84\pi.
\end{aligned}$$

We now calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$. *See Figure* **??**.

Figure 2: *Question 2.*

On S_1 $(z = 0)$: $\widehat{\mathbf{n}} = -\mathbf{k}$. *So*

$$\mathbf{F} \cdot \widehat{\mathbf{n}} = (4x, -2y^2, z^2) \cdot (0, 0, -1) = -z^2 = 0,$$

since $z = 0$. *So* $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = 0$.

On S_2 *(*$z=3$*):* $\widehat{\mathbf{n}}=\mathbf{k}$*. So*

$$\mathbf{F}\cdot\widehat{\mathbf{n}}=(4x,-2y^2,z^2)\cdot(0,0,1)=z^2=9,$$

since $z=3$*. So*

$$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}=\iint_{S_2}\mathbf{F}\cdot\widehat{\mathbf{n}}\,dS=9\iint_{S_2}dS=36\pi,$$

since the area of S_2 *is* 4π*.*

Finally, on S_3 *(*$x^2+y^2=4$*): Parameterise cylinder as*

$$x=2\cos\theta,\quad y=2\sin\theta,\quad z=z,\qquad 0\le\theta\le 2\pi,\quad 0\le z\le 3.$$

So $\mathbf{r}(\theta,z)=(2\cos\theta,2\sin\theta,z)$*. Also*

$$\mathbf{r}_\theta\times\mathbf{r}_z=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2\sin\theta&2\cos\theta&0\\0&0&1\end{vmatrix}=(2\cos\theta,2\sin\theta,0),$$

and

$$\mathbf{F}(\mathbf{r}(\theta,z))=(8\cos\theta,-8\sin^2\theta,z^2).$$

So

$$\begin{aligned}\iint_{S_3}\mathbf{F}\cdot d\mathbf{S}&=\iint_R\mathbf{F}(\mathbf{r}(\theta,z))\cdot(\mathbf{r}_\theta\times\mathbf{r}_z)\,d\theta\,dz\\&=\int_0^3\int_0^{2\pi}\left(16\cos^2\theta-16\sin^3\theta\right)d\theta\,dz\\&=48\int_0^{2\pi}\left(\cos^2\theta-\sin^3\theta\right)d\theta=48\pi.\end{aligned}$$

Thus

$$\iint_S\mathbf{F}\cdot d\mathbf{S}=\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}+\iint_{S_2}\mathbf{F}\cdot d\mathbf{S}+\iint_{S_3}\mathbf{F}\cdot d\mathbf{S}=36\pi+48\pi=84\pi.$$

3. Verify Stokes' theorem for the vector field $\mathbf{F}=z\mathbf{i}-3x\mathbf{j}+2z\mathbf{k}$, where S is the surface $z=1-x^2-y^2$, $z\ge 0$, C is the boundary circle $x^2+y^2=1$. Assume that S is oriented in the positive z-direction.

Solution. Stokes' Theorem states $\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}$*. See Figure* **??***.*

Figure 3: *Question 3.*

Parameterise C *as*

$$x = \cos\theta, \quad y = \sin\theta, \quad z = 0, \qquad 0 \le \theta \le 2\pi.$$

Thus $\mathbf{r}(\theta) = (\cos\theta, \sin\theta, 0)$ *and so*

$$\mathbf{r}'(\theta) = (-\sin\theta, \cos\theta, 0), \qquad \mathbf{F}(\mathbf{r}(\theta)) = (0, -3\cos\theta, 0).$$

Hence

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)\,d\theta d = \int_0^{2\pi} (-3\cos^2\theta)\,d\theta = -3\pi.$$

Now

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -3x & 2y \end{vmatrix} = (2, 1, -3),$$

and so

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_S (2, 1, -3)\cdot d\mathbf{S}.$$

Parameterise S *as*

$$x = x, \quad y = y, \quad z = 1 - x^2 - y^2 \qquad \Longrightarrow \qquad \mathbf{r}(x, y) = (x, y, 1 - x^2 - y^2).$$

So

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x, 2y, 1).$$

Thus

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_R (\nabla\times\mathbf{F})(x,y)\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dx\,dy = \iint_R (4x + 2y - 3)\,dx\,dy.$$

R *is the disc* $x^2 + y^2 = 1$ *and so let*

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

Hence

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} == \int_0^{2\pi}\int_0^1 \big(4r\cos\theta + 2r\sin\theta - 3\big) r\, dr\, d\theta = \int_0^{2\pi}\left(\frac{4}{3}\cos\theta + \frac{2}{3}\sin\theta - \frac{3}{2}\right) d\theta = -3\pi.$$

4. For each of the following equations, state the order and whether it is linear or nonlinear, and homogeneous or inhomogeneous:

$$(a) \qquad u_{tt} - u_{xx} + x^2 = 0$$

$$(b) \qquad u_t - u_{xx} + u/x = 0$$

$$(c) \qquad u_x(1+u^2)^{1/2} + u_y(1+u_y^2)^{-1/2} = 0$$

$$(d) \qquad u_t - u_{xx} + xu = 0$$

Solution (a). Second order, linear, inhomogeneous.

Solution (b). Second order, linear, homogeneous.

Solution (c). First order, non-linear, homogeneous.

Solution (d). Second order, linear, homogeneous.

5. Solve the following psuedo (or degenerate) PDEs assuming $u = u(x,y)$.

$$(a) \qquad u_x + u = 0$$

$$(b) \qquad u_x - yu = 0$$

$$(c) \qquad u_x + x^2 u = 0$$

$$(d) \qquad u_{xx} + u_x + u = 0$$

Solution (a). Consider the ODE

$$\frac{du}{dx} + u = 0.$$

This is separable and so

$$\frac{du}{u} = -dx \qquad \Longrightarrow \qquad \ln u = -x + A,$$

or $u = Ce^{-x}$*. Hence the solution of the PDE is* $u(x,y) = C(y)e^{-x}$*.*

Solution (b). Consider the ODE

$$\frac{du}{dx} - yu = 0.$$

This is separable and so

$$\frac{du}{u} = y dx \qquad \Longrightarrow \qquad \ln u = yx + A,$$

or $u = Ce^{xy}$. *Hence the solution of the PDE is* $u(x,y) = C(y)e^{xy}$.

Solution (c). Consider the ODE

$$\frac{du}{dx} + x^2 u = 0.$$

Again this is separable and so

$$\frac{du}{u} = -x^2 dx \qquad \Longrightarrow \qquad \ln u = -\frac{1}{3}x^3 + A,$$

or $u = Ce^{-x^3/3}$. *Hence the solution of the PDE is* $u(x,y) = C(y)e^{-x^3/3}$.

Solution (d). Consider the ODE

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + u = 0.$$

Let $u = e^{\lambda x}$. Then the ODE becomes

$$e^{\lambda x}\big(\lambda^2 + \lambda + 1\big) = 0 \qquad \Longrightarrow \qquad p^2 + p + 1 = 0 \qquad \Longrightarrow \qquad p = \frac{-1 + \sqrt{3}i}{2}.$$

Thus

$$\begin{aligned} u &= Ae^{-x/2+\sqrt{3}ix/2} + Be^{-x/2-\sqrt{3}ix/2} \\ &= e^{-x/2}\big[Ae^{\sqrt{3}ix/2} + Be^{-\sqrt{3}ix/2}\big] \\ &= e^{-x/2}\big[C\cos(\sqrt{3}x/2) + D\sin(\sqrt{3}x/2)\big], \end{aligned}$$

and so

$$u(x,y) = e^{-x/2}\big[C(y)\cos(\sqrt{3}x/2) + D(y)\sin(\sqrt{3}x/2)\big].$$

6. Fully classify the following equations and find and sketch the characteristics (if any) of the following partial differential equations.

(a) $\quad u_t - u_{xx} + 1 = 0$

(b) $\quad u_{tt} + 2u_{xt} - u_{xx} - u_t + u_x = 0$

(c) $\quad u_{xx} + 5u_{yy} = 0$

(d) $\quad 4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$

Solution. Characteristics satisfy

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

(a) $A = -1$, $B = C = 0$. *Thus* $B^2 = AC = 0$ *and so the equation is parabolic. Hence there is* **one** *characteristic which satisfies*

$$\frac{dt}{dx} = \frac{0 \pm \sqrt{0}}{-1} = 0 \qquad \Longrightarrow \qquad t = c.$$

Hence the characteristics are horizontal straight lines, see Figure **??**.

(b) $A = -1$, $2B = 2$, $C = 1$. *Thus* $B^2 - AC = 1 - (-1)(1) = 2 > 0$ *and so the equation is hyperbolic. Hence there are* **two** *characteristics which satisfy*

$$\frac{dt}{dx} = \frac{1 \pm \sqrt{2}}{-1} = -1 \pm \sqrt{2} \qquad \Longrightarrow \qquad t = (-1+\sqrt{2})x + c \quad or \quad t = (-1-\sqrt{2})x + c.$$

The two sets of characteristics are given in Figure **??**.

(c) $A = 1$, $B =$, $C = 5$. *Thus* $B^2 - AC = 0 - (1)(5) = -5 < 0$ *and so the equation is elliptic. Hence there are* **no** *characteristics.*

(d) $A = 4$, $2B = -12$, $C = 9$. *Thus* $B^2 - AC = (-6)^2 - (4)(9) = 0$ *and so the equation is parabolic. Hence there is* **one** *characteristic which satisfies*

$$\frac{dy}{dx} = \frac{-6 \pm \sqrt{0}}{4} = -\frac{3}{2} \qquad \Longrightarrow \qquad y = -\frac{3}{2}x + c.$$

The characteristics are given in Figure **??**.

Figure 4: *Question 6.*

7. Check whether $u(x,y) = \sinh x \cosh y$ and $u(x,y) = x^3 - 3xy^2 + 6x^2y - 2y^3$ are solutions to the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Solution. Consider $u = \sinh x \cosh y$. *Then*

$$u_x = \cosh x\, \cosh y, \quad u_{xx} = \sinh x\, \cosh y, \quad u_y = \sinh x\, \sinh y, \quad u_{yy} = \sinh x\, \cosh y.$$

Hence

$$u_{xx} + u_{yy} = 2 \sinh x\, \cosh y \neq 0,$$

and so $u = \sinh x \cosh y$ *is not a solution.*

If $u = x^3 - 3xy^2 + 6x^2y - 2y^3$ *then*

$$u_x = 3x^2 - 3y^2 + 12xy, \quad u_{xx} = 6x + 12y, \quad u_y = -6xy + 6x^2 - 6y^2, \quad u_{yy} = -6x - 12y.$$

Thus

$$u_{xx} + u_{yy} = 6x + 12y - 6x - 12y = 0,$$

and so u *is a solution of Laplace's equation.*

8. Consider the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0.$$

Use the transformation $\alpha = x - 2y$, $\beta = x - y$ to reduce the equation to a simpler form and determine the general solution $u(x, y)$.

Solution. Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta},$$

and so $u_x = u_\alpha + u_\beta$. *This means*

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right)\frac{\partial \beta}{\partial x} \\ &= \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2},\end{aligned}$$

and so $u_{xx} = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}$. *Also*

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial y} = -2\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta},$$

and so $u_y = -2u_\alpha - u_\beta$. *This means*

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial \alpha}\left(-2\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(-2\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \beta}{\partial y} \\ &= 4\frac{\partial^2 u}{\partial \alpha^2} + 4\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2},\end{aligned}$$

and so $u_{yy} = 4u_{\alpha\alpha} + 4u_{\alpha\beta} + u_{\beta\beta}$. *Also*

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial \alpha}\left(-2\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(-2\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \beta}{\partial x} \\ &= -2\frac{\partial^2 u}{\partial \alpha^2} - 3\frac{\partial^2 u}{\partial \alpha \partial \beta} - \frac{\partial^2 u}{\partial \beta^2},\end{aligned}$$

and so $u_{xy} = -2u_{\alpha\alpha} - 3u_{\alpha\beta} - u_{\beta\beta}$. *Thus* $u_{xx} + 2u_{xy} + u_{yy} = 0$ *becomes*

$$\begin{aligned} & u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta} + 2(-2u_{\alpha\alpha} - 3u_{\alpha\beta} - u_{\beta\beta}) + 4u_{\alpha\alpha} + 4u_{\alpha\beta} + u_{\beta\beta} = 0 \\ \Longrightarrow \quad & u_{\alpha\alpha} = 0 \\ \Longrightarrow \quad & u_{\alpha} = f(\beta) \\ \Longrightarrow \quad & u = f(\beta)\alpha + g(\beta) \\ \Longrightarrow \quad & u(x,y) = f(x-y)(x-2y) + g(x-y). \end{aligned}$$

9. Classify the PDE

$$u_{xx} + 2ku_{xy} + k^2 u_{yy} = 0, \quad k \neq 0.$$

Using a suitable transformation of form $\xi = x + ay$; $\eta = x + by$, show that the equation can be transformed into the form

$$u_{\xi\xi} = 0\ ,$$

and hence find the solution of the equation in terms of two arbitrary functions.

Solution. Here $A = 1$, $2B = 2k$ *and* $C = k^2$. *Thus* $B^2 - AC = k^2 - k^2 = 0$ *and so the PDE is parabolic. Let* $\xi = x + ay$ *and* $\eta = x + by$. *Then*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

and so $u_x = u_\xi + u_\eta$. *Also*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

or $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$. *Now*

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta},$$

or $u_y = au_\xi + bu_\eta$. *Also*

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \xi}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y} = a^2\frac{\partial^2 u}{\partial \xi^2} + 2ab\frac{\partial^2 u}{\partial \xi \partial \eta} + b^2\frac{\partial^2 u}{\partial \eta^2},$$

or $u_{yy} = a^2 u_{\xi\xi} + 2ab u_{\xi\eta} + b^2 u_{\eta\eta}$. *Finally*

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial \xi}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = a\frac{\partial^2 u}{\partial \xi^2} + (a+b)\frac{\partial^2 u}{\partial \xi \partial \eta} + b\frac{\partial^2 u}{\partial \eta^2},$$

or $u_{xy} = au_{\xi\xi} + (a+b)u_{\xi\eta} + bu_{\eta\eta}$. *Thus* $u_{xx} + 2ku_{xy} + k^2u_{yy} = 0$ *becomes*

$$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} + 2k\big(au_{\xi\xi} + (a+b)u_{\xi\eta} + bu_{\eta\eta}\big) + k^2\big(a^2u_{\xi\xi} + 2abu_{\xi\eta} + b^2u_{\eta\eta}\big) = 0$$

$$\implies \qquad (1+ka)^2u_{\xi\xi} + 2(1+ka)(1+kb)u_{\xi\eta} + (1+kb)^2u_{\eta\eta} = 0.$$

We want $u_{\xi\xi} = 0$ *and so we choose* $b = -1/k$. *Then provided* $a \neq -1/k$ *we have* $u_{\xi\xi} = 0$. *Thus*

$$u_\xi = f(\eta) \qquad \implies \qquad u = f(\eta)\xi + g(\eta).$$

Finally this gives

$$u(x,y) = f(x+by)(x+ay) + g(x+by).$$

10. Solve $u_t = c^2u_{xx}$ using the method of separation of variables subject to the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$ and the initial condition $u(x,0) = x(2-x)$.

Solution. Let $u(x,t) = F(x)G(t)$. *Then*

$$u_t = F(x)\dot{G}(t) \qquad \& \qquad u_{xx} = F''(x)G(t).$$

Substituting these into $u_t = c^2u_{xx}$ *gives*

$$F(x)\dot{G}(t) = c^2F''(x)G(t) \qquad \implies \qquad \frac{\dot{G}}{c^2G} = \frac{F''}{F} = k,$$

or

$$\dot{G} - c^2kG = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u(0,t) = u(1,t) = 0$ *therefore imply* $F(0) = F(1) = 0$.

- $k = 0$:

  $$F'' = 0 \qquad \implies \qquad F(x) = Ax + B.$$

  $F(0) = 0$ *implies that* $B = 0$ *and* $F(1) = 0$ *implies that* $A = 0$. *Hence there are no non-trivial solutions.*

- $k = p^2 > 0$: *The solution of* $F'' - p^2F = 0$ *is*

  $$F(x) = Ae^{px} + Be^{-px}.$$

  The bc $F(0) = 0$ *implies* $A + B = 0$ *and the bc* $F(1) = 0$ *implies* $Ae^p + Be^{-p} = 0$. *Then* $A = B = 0$ *and so there are no non-trivial solutions.*

- *$k=-p^2<0$: The solution of $F''+p^2F=0$ is*

$$F(x) = A\cos px + B\sin px.$$

The bc $F(0)=0$ implies $A=0$ and the bc $F(1)=0$ implies $B\sin p=0$, and so $p=n\pi$. Thus

$$F_n(x) = B_n \sin n\pi x.$$

Now let's look at the ODE for G. We have

$$\dot{G} + c^2n^2\pi^2 G_n = 0 \qquad \Longrightarrow \qquad G_n(t) = C_n e^{-\lambda_n^2 t},$$

where $\lambda_n = cn\pi$. Hence

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin n\pi x\, C_n e^{-\lambda_n^2 t},$$

and so

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin n\pi x\, e^{-\lambda_n^2 t},$$

where $D_n = B_nC_n$. Finally, to find D_n we use the initial condition $u(x,0)=x(2-x)$. Then

$$x(2-x) = \sum_{n=1}^{\infty} D_n \sin n\pi x\ ,$$

and so

$$\begin{aligned}
D_n &= 2\int_0^1 x(2-x)\sin n\pi x\, dx \\
&= 4\int_0^1 x\sin n\pi x\, dx - 2\int_0^1 x^2 \sin n\pi x\, dx \\
&= 4\left[\frac{\sin n\pi - n\pi\cos n\pi}{n^2\pi^2}\right] - 2\left[\frac{-n^2\pi^2\cos n\pi + 2\cos n\pi + 2n\pi\sin n\pi - 2}{n^3\pi^3}\right] \\
&= -\frac{4}{n\pi}(-1)^n - 2\left(-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right) \\
&= -\frac{2(-1)^n}{n\pi} - \frac{4(-1)^n}{n^3\pi^3} + \frac{4}{n^3\pi^3}.
\end{aligned}$$