MA4006: Exercise Sheet 3: Solutions

1. Evaluate the integral $\iint_R dx\, dy$ over the triangle with vertices $(-1,0)$, $(0,2)$ and $(2,0)$.

Solution. See Figure 1. Let x be the inner variable and y the outer variable. Thus we need the equations of L_1 and L_2. Use $y = mx + c$ and substitute in points $(-1,0)$, $(0,2)$. We find that $m = c = 2$ and so $y = 2x + 2$ or $x = (y-2)/2$. Similarly, L_2 is the line $y = -x + 2$ and so $x = -y + 2$. Hence

$$\iint_R dx\, dy = \int_0^2 \int_{(y-2)/2}^{-y+2} dx\, dy = \int_0^2 \left(-\frac{3}{2}y + 3\right) dy = 3.$$

Note that this is precisely the area of the triangle.

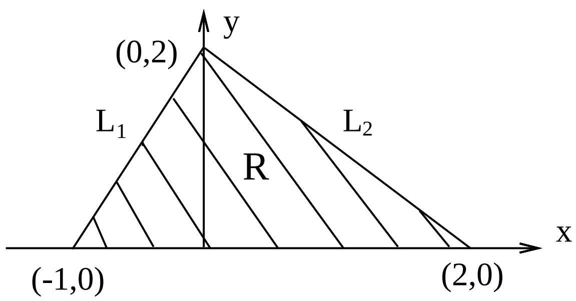

Figure 1: *Question 1.*

2. Evaluate the integral $\iint_R y^{-1/2}\, dx\, dy$ over the area bounded by $y = x^2$, $x + y = 2$, and the y-axis.

Solution. See Figure 2. Let y be the inner variable and x the outer variable. Then

$$\begin{aligned}\iint_R y^{-1/2}\, dx\, dy &= \int_0^1 \int_{x^2}^{2-x} y^{-1/2}\, dy\, dx \\ &= \int_0^1 \left[2y^{1/2}\right]_{x^2}^{2-x} dx \\ &= \int_0^1 \left(2(2-x)^{1/2} - 2x\right) dx = \left[-\frac{4}{3}(2-x)^{3/2} - x^2\right]_0^1 \approx 1.437867.\end{aligned}$$

3. Evaluate $\int_C \Omega\, ds$ where $\Omega(x,y) = x^2 + y^2$ and C is the segment of the line $y = 3x$ from $(0,0)$ to $(2,6)$.

Solution. Let $x = t$ and then $y = 3x = 3t$. The point $(0,0)$ is $t = 0$ and the point $(2,6)$ is

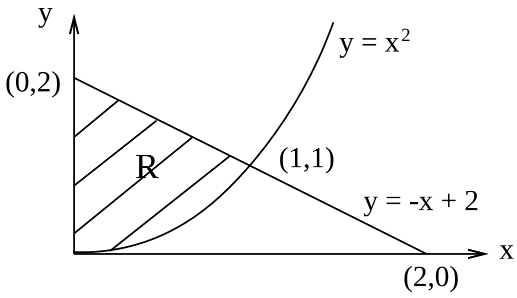

Figure 2: *Question 2.*

$t = 2$. *Thus*

$$\int_C \Omega\, ds = \int_0^2 \Omega(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt.$$

Now, $\mathbf{r}(t) = (t, 3t)$ *and so* $\mathbf{r}'(t) = (1, 3)$, $|\mathbf{r}'(t)| = \sqrt{10}$. *Also* $\Omega(\mathbf{r}(t)) = x^2 + y^2 = t^2 + 9t^2 = 10t^2$. *So*

$$\int_C \Omega\, ds = \int_0^2 10t^2\sqrt{10}\, dt = \frac{80\sqrt{10}}{3}.$$

4. Evaluate $\int_C f\, ds$ where $f(x, y, z) = 1 + y^2 + z^2$ and $C : \mathbf{r} = (t, \cos t, \sin t)$, $0 \le t \le 2\pi$.
Solution. Here $\mathbf{r}'(t) = (1, -\sin t, \cos t)$ *and so* $|\mathbf{r}'(t)| = \sqrt{2}$. *Also* $f(\mathbf{r}(t)) = 1 + \cos^2 t + \sin^2 t = 2$. *Thus*

$$\int_C f\, ds = \int_0^{2\pi} f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt = \int_0^{2\pi} 2\sqrt{2}\, dt = 4\sqrt{2}\pi.$$

5. Find the work done in moving a particle in a force field given by

$$\mathbf{F} = 3xy\mathbf{i} - y^2\mathbf{j} + 10x\mathbf{k}$$

along the curve $y = 2x^2$, $z = 0$, from the point $(0, 0, 0)$ to the point $(1, 2, 0)$.
Solution. The work done is $\int_C \mathbf{F} \cdot d\mathbf{r}$. *To find the parametric definition of* C, *we set* $x = t$ *and then* $y = 2x^2 = 2t^2$, *with* $z = 0$. *The points* $(0, 0, 0)$ *and* $(1, 1, 0)$ *correspond to* $t = 0$ *and* $t = 1$ *respectively. So* $\mathbf{r}(t) = (t, 2t^2, 0)$, $0 \le t \le 1$, *giving* $\mathbf{r}'(t) = (1, 4t, 0)$ *and*

$$\mathbf{F}(\mathbf{r}(t)) = \big(3(t)(2t^2), -(2t^2)^2, 10(t)\big) = (6t^3, -4t^4, 10t).$$

Thus

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_0^1 (6t^3, -4t^4, 10t)\cdot(1,4t,0)\,dt = \int_0^1 (6t^3+16t^5)\,dt = \frac{7}{6}.$$

6. If $\mathbf{f} = 8x^2yz\mathbf{i}+5z\mathbf{j}-4xy\mathbf{k}$, find the work done in moving a particle in a force field given by $\mathbf{f}$ along the curve C with parametric definition $\mathbf{r}(t) = (t, t^2, t^3)$, $(0 \le t \le 1)$.

Solution. Here $\mathbf{r}'(t) = (1, 2t, 3t^2)$ *and*

$$\mathbf{f}(\mathbf{r}(t)) = \big(8(t)^2(t^2)(t^3), 5(t^3), -4(t)(t^2)\big) = (8t^7, 5t^3, -4t^3).$$

Thus,

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_0^1 \mathbf{f}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (8t^7, 5t^3, -4t^3)\cdot(1,2t,3t^2)\,dt = \int_0^1 (8t^7+10t^4-12t^5)\,dt = 1.$$

7. Show that $\mathbf{f} = y^2\mathbf{i}+2xy\mathbf{j}$ is a conservative vector field. Determine and associated *scalar potential* ϕ for this vector field $\mathbf{f}$.

Solution.

$$\nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & 0 \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(2y-2y) = \mathbf{0}.$$

Hence $\mathbf{f}$ *is conservative and so there exists* ϕ *such that* $\mathbf{f} = \nabla\phi$, *i.e.*

$$(y^2, 2xy, 0) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

or

$$\text{(i)}\quad \frac{\partial\phi}{\partial x} = y^2, \qquad \text{(ii)}\quad \frac{\partial\phi}{\partial y} = 2xy, \qquad \text{(iii)}\quad \frac{\partial\phi}{\partial z} = 0.$$

(i) $\frac{\partial\phi}{\partial x} = y^2$ *and so*

$$\phi = xy^2 + g(y,z) \qquad \Longrightarrow \qquad \frac{\partial\phi}{\partial y} = 2xy + \frac{\partial g}{\partial y}. \tag{1}$$

(ii) Also $\frac{\partial\phi}{\partial y} = 2xy$ *and so comparing with (1) we see that* $\frac{\partial g}{\partial y} = 0$. *So* $g = g(z)$ *and* $\phi = xy^2 + g(z)$.

(iii) Finally, $\frac{\partial \phi}{\partial z} = 0$ *which means that* $\frac{\partial g}{\partial z} = 0$ *and so* g *is constant.*

Thus the scalar potential is $\phi(x, y, z) = xy^2 + c$ *where* c *is an arbitrary constant.*

8. If $\mathbf{a} = (3x^2 + 6y, -14yz, 20xz^2)$ evaluate $\int_C \mathbf{a} \cdot d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the following paths:

(a) The straight lines from $(0, 0, 0)$ to $(1, 0, 0)$, then to $(1, 1, 0)$ and finally to $(1, 1, 1)$.

(b) The straight line from $(0, 0, 0)$ to $(1, 1, 1)$.

Solution (a). The line from $(0, 0, 0)$ *to* $(1, 0, 0)$ *is parameterised by*

$$C_1 : \qquad \mathbf{r}(t) = (t, 0, 0), \qquad 0 \leq t \leq 1.$$

Thus

$$\mathbf{a}(\mathbf{r}(t)) = (3t^2, 0, 0) \qquad \& \qquad \mathbf{r}'(t) = (1, 0, 0),$$

and so

$$I_1 = \int_0^1 \mathbf{a}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 3t^2\, dt = 1.$$

The line from $(1, 0, 0)$ *to* $(1, 1, 0)$ *is parameterised by*

$$C_2 : \qquad \mathbf{r}(t) = (1, t, 0), \qquad 0 \leq t \leq 1.$$

Thus

$$\mathbf{a}(\mathbf{r}(t)) = (3 + 6t, 0, 0) \qquad \& \qquad \mathbf{r}'(t) = (0, 1, 0),$$

and so

$$I_2 = \int_0^1 \mathbf{a}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 0\, dt = 0.$$

Finally, the line from $(1, 1, 0)$ *to* $(1, 1, 1)$ *is parameterised by*

$$C_3 : \qquad \mathbf{r}(t) = (1, 1, t), \qquad 0 \leq t \leq 1.$$

Thus

$$\mathbf{a}(\mathbf{r}(t)) = (9, -14t, 20t^2) \qquad \& \qquad \mathbf{r}'(t) = (0, 0, 1),$$

and so

$$I_3 = \int_0^1 \mathbf{a}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 20t^2\, dt = \frac{20}{3}.$$

Hence $\int_C \mathbf{a} \cdot d\mathbf{r} = I_1 + I_2 + I_3 = \frac{23}{3}$.

Solution (b). The line from $(0,0,0)$ *to* $(1,1,1)$ *is parameterised by*

$$C: \qquad \mathbf{r}(t) = (t,t,t), \qquad 0 \le t \le 1.$$

Thus

$$\mathbf{a}(\mathbf{r}(t)) = (3t^2+6t, -14t^2, 20t^3) \qquad \& \qquad \mathbf{r}'(t) = (1,1,1),$$

and so

$$\int_C \mathbf{a}\cdot d\mathbf{r} = \int_0^1 \mathbf{a}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (3t^2+6t-14t^2+20t^3)\,dt = \frac{13}{3}.$$

9. Let $\phi = 4x$ and let V denote the closed region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$ and $z=0$. Evaluate $\iiint_V \phi\,dV$.

Solution. See Figure 3. If z *is the inner variable then* z *varies from 0 to* $z = 8-4x-2y$.

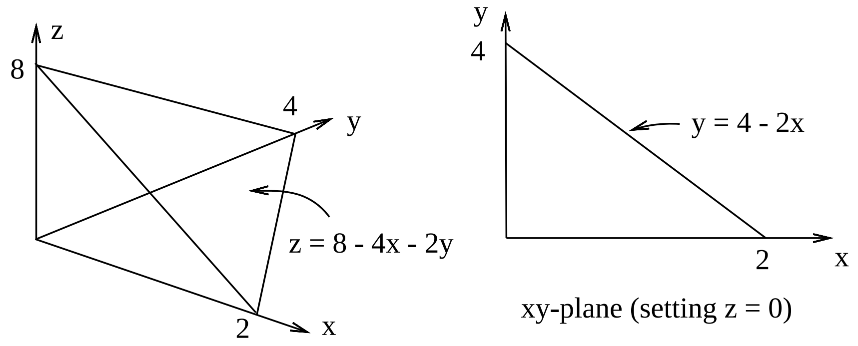

Figure 3: *Question 9.*

If y *is the middle variable then (in the xy-plane, setting* $z=0$*, see Figure 3)* y *varies from 0 to* $4-2x$*. If* x *is the outer variable then* x *varies from 0 to 2. Thus*

$$\begin{aligned}\iiint_V \phi\,dV = \int_0^2\int_0^{4-2x}\int_0^{8-4x-2y} 4x\,dz\,dy\,dx &= \int_0^2\int_0^{4-2x} 4x(8-4x-2y)\,dy\,dx \\ &= \int_0^2\int_0^{4-2x} (32x-16x^2-8xy)\,dz\,dy\,dx \\ &= \int_0^2 (64x-64x^2+16x^3)\,dx = \frac{64}{3}.\end{aligned}$$

10. Using Green's Theorem, evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ counterclockwise around the boundary of C of the region R, where $\mathbf{f} = (\sin y, \cos x)$ and R is the triangle with vertices $(0, 0)$, $(\pi, 0)$ and $(\pi, 1)$.

Solution. Green's Theorem says

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\, dy.$$

Now $\mathbf{r} = (\sin y, \cos x)$ *and* $\frac{\partial f_1}{\partial y} = \cos y$ *and* $\frac{\partial f_2}{\partial x} = -\sin x$*. The Region* R *and boundary curve* C *are shown in Figure 4. Thus*

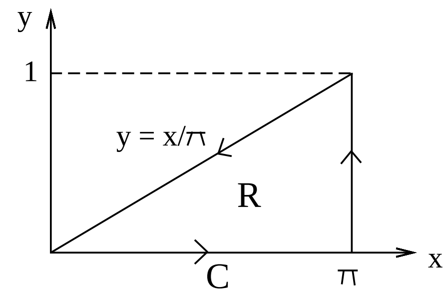

Figure 4: *Question 10.*

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (-\sin x - \cos y)\, dx\, dy &= \int_0^{\pi} \int_0^{x/\pi} (-\sin x - \cos y)\, dy\, dx \\
&= \int_0^{\pi} \big[-y \sin x - \sin y \big]_0^{x/\pi} dx \\
&= \int_0^{\pi} \left(-\frac{x}{\pi} \sin x - \sin \frac{x}{\pi} \right) dx = \pi \cos 1 - 1 - \pi.
\end{aligned}$$

11. Evaluate $\iint_S \mathbf{a} \cdot \hat{\mathbf{n}}\, dS \equiv \iint_S \mathbf{a} \cdot d\mathbf{S}$ where $\mathbf{a} = (18z, -12, 3y)$ and S is part of the plane $2x + 3y + 6z = 12$ located in the first octant.

Solution. See Figure 5. Plane $2x + 3y + 6z = 12$ *is*

$$z = 2 - \frac{x}{3} - \frac{y}{2} = g(x, y).$$

Let $\mathbf{r}(u, v) = \mathbf{r}(x, y) = (x, y, g(x, y))$*. So*

$$\mathbf{r}_x = (1, 0, g_x) = (1, 0, -1/3), \qquad \mathbf{r}_y = (0, 1, g_y) = (0, 1, -1/2).$$

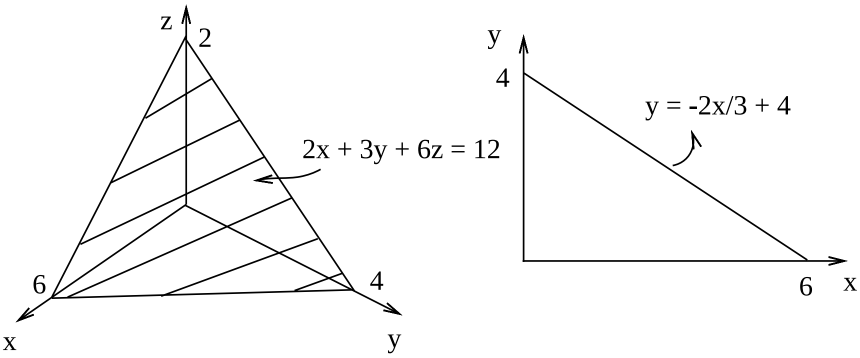

Figure 5: *Question 11.*

Thus

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \end{vmatrix} = (1/3, 1/2, 1).$$

Also

$$\mathbf{a}(\mathbf{r}(x, y)) = (36 - 6x - 9y, -12, 3y).$$

So

$$\begin{aligned} \iint_S \mathbf{a} \cdot d\mathbf{S} = \iint_R \mathbf{a} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dx\, dy &= \int_0^6 \int_0^{4-2x/3} (36 - 6x - 9y, -12, 3y) \cdot (1/3, 1/2, 1)\, dy\, dx \\ &= \int_0^6 \left[(6 - 2x)y\right]_0^{4-2x/3} dx \\ &= \int_0^6 \left(\frac{4}{3}x^2 - 12x + 24\right) dx = 24. \end{aligned}$$

12. Use the Divergence Theorem to compute the surface integral $\iint_S \mathbf{a} \cdot \mathbf{dS}$ where $\mathbf{a} = (4x, -2y^2, z^2)$ and S is the cylindrical surface $x^2 + y^2 = 4$, $0 \le z \le 3$.

Solution. The Divergence Theorem states that $\iint_S \mathbf{a} \cdot \mathbf{dS} = \iiint_V \nabla \cdot \mathbf{a}\, dV$. *Now*

$$\nabla \cdot \mathbf{a} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (4x, -2y^2, z^2) = 4 - 4y + 2z.$$

Use cylindrical co-ordinates

$$\mathbf{r} = (r\cos\theta, r\sin\theta, z), \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 3.$$

Then

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r, \qquad (\nabla\cdot\mathbf{a})(\mathbf{r}(r,\theta,z)) = 4 - 4r\sin\theta + 2z.$$

Thus

$$\begin{aligned}\iiint_V \nabla\cdot\mathbf{a}\,dV &= \int_0^3\int_0^{2\pi}\int_0^2 (4-4r\sin\theta+2z)r\,dr\,d\theta\,dz\\ &= \int_0^3\int_0^{2\pi}\int_0^2 (4r-4r^2\sin\theta+2zr)\,dr\,d\theta\,dz\\ &= \int_0^3\int_0^{2\pi}\left(8-\frac{32}{3}\sin\theta+4z\right)\,d\theta\,dz\\ &= \int_0^3 (16\pi+8\pi z)\,dz = 84\pi.\end{aligned}$$

13. Verify Stokes' theorem for the case $\mathbf{f} = (2x-y, -yz^2, -y^2z)$ where the surface S is the hemisphere $x^2+y^2+z^2=1$, $z\geq 0$ and the curve C is defined as the unit circle.

Solution. See Figure 6. Stokes' Theorem: $\oint_C \mathbf{f}\cdot d\mathbf{r} = \iint_S(\nabla\times\mathbf{f})\cdot d\mathbf{S}$. *First let us find*

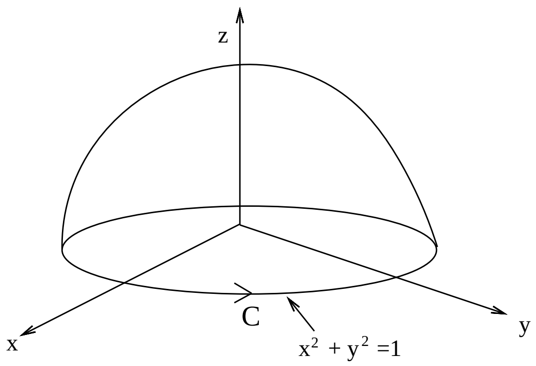

Figure 6: *Question 11.*

$\oint_C \mathbf{f}\cdot d\mathbf{r}$. *Since C is the unit circle we have* $\mathbf{r}(t) = (\cos t, \sin t, 0)$, $0\leq t\leq 2\pi$. *Then*

$\mathbf{r}'(t) = (-\sin t, \cos t, 0)$ *and* $\mathbf{f}(\mathbf{r}(t)) = (2\cos t - \sin t, 0, 0)$. *Thus*

$$\begin{aligned}
\oint_C \mathbf{f} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} (2\cos t - \sin t, 0, 0) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_0^{2\pi} (-2\sin t \cos t + \sin^2 t)\, dt \\
&= \int_0^{2\pi} (-\sin 2t + \sin^2 t)\, dt \\
&= 0 + \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\, dt = \pi.
\end{aligned}$$

Now find $\iint_S (\nabla \times \mathbf{f}) \cdot d\mathbf{S}$. *We have*

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = (0, 0, 1).$$

The surface of hemisphere is given parametrically by

$$\mathbf{r}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad 0 \le \theta \le \pi/2, \quad 0 \le \phi \le 2\pi,$$

and

$$\begin{aligned}
\mathbf{r}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\
\mathbf{r}_\phi &= (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) \\
\mathbf{r}_\theta \times \mathbf{r}_\phi &= (\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta).
\end{aligned}$$

Hence

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{f}) \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi/2} (0, 0, 1) \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi)\, d\theta\, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta\, d\phi \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^{\pi/2} \sin 2\theta\, d\theta\, d\phi \\
&= \frac{1}{2} \int_0^{2\pi} d\phi = \pi.
\end{aligned}$$

We have therefore verified Stokes' Theorem.