MA4006: Exercise Sheet 2: Solutions

1. If $f(x,y) = ax^2 + 2hxy + by^2$, where a, b and h are all constants, find all the first and second order partial derivatives and demonstrate that $f_{xy} = f_{yx}$. *Solution.*

$$f_x = 2ax + 2hy, \quad f_y = 2hx + 2by, \quad f_{xx} = 2a, \quad f_{yy} = 2b\ , \quad f_{xy} = 2h, \quad f_{yx} = 2h.$$

Hence $f_{xy} = f_{yx} = 2h$.

2. If $u = (Ar^n + Br^{-n})\cos(n\theta)$, where A, B, n are constants, show that

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Solution.

$$\begin{aligned}
u_r &= (nAr^{n-1} - nBr^{-n-1})\cos n\theta \\
u_{rr} &= [n(n-1)Ar^{n-2} + n(n+1)Br^{-n-2}]\cos n\theta \\
u_\theta &= (Ar^n + Br^{-n})(-n\sin n\theta) = -(nAr^n + nBr^{-n})\sin n\theta \\
u_{\theta\theta} &= -(n^2Ar^n + n^2Br^{-n})\cos n\theta.
\end{aligned}$$

So

$$\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\
&= [n(n-1)Ar^{n-2} + n(n+1)Br^{-n-2}]\cos n\theta \\
&\quad + \frac{1}{r}(nAr^{n-1} - nBr^{-n-1})\cos n\theta - \frac{1}{r^2}(n^2Ar^n + n^2Br^{-n})\cos n\theta \\
&= [n^2 - n + n - n^2]Ar^{n-2}\cos n\theta + [n^2 + n - n - n^2]Br^{n-2}\cos n\theta = 0.
\end{aligned}$$

3. Find the directional derivative of

$$f(x,y,z) = 3xy + 2z^2 + 4x,$$

at the point $P(1,0,-2)$ in the direction of the vector $\mathbf{a} = \mathbf{j} - 2\mathbf{k}$.

Solution. We need a unit vector in the direction of $\mathbf{a} = \mathbf{j} - 2\mathbf{k}$, *i.e.*

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{5}}(\mathbf{j} - 2\mathbf{k}).$$

Also,

$$\nabla f = (3y+4)\mathbf{i} + 3x\mathbf{j} + 4z\mathbf{k} \quad \Rightarrow \quad \nabla f|_{P(1,0,-2)} = 4\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}.$$

So

$$\frac{\partial f}{\partial n} = (4, 3, -8) \cdot \frac{1}{\sqrt{5}}(0, 1, -2) = \frac{19}{\sqrt{5}}.$$

4. Explain the term *level surfaces* in relation to any scalar field ω. Interpret $\nabla\omega$ with regard to level surfaces. Hence, find the unit normal to the surface defined by $z = \sqrt{2xy + x^2}$, at the point $(2, 1, 2)$.

Solution. The level surfaces of $\omega(x, y, z)$ are those surfaces defined by

$$w(x, y, z) = c, \qquad c \in \mathbf{R}.$$

∇w is normal to the level surface $w(x, y, z) = c$. Consider $z = \sqrt{2xy + x^2}$. We can write this as $w(x, y, z) = 0$ where

$$w(x, y, z) = z - \sqrt{2xy + x^2}.$$

Then

$$\nabla w = -\frac{(2y + 2x)}{2\sqrt{2xy + x^2}}\mathbf{i} - \frac{2x}{2\sqrt{2xy + x^2}}\mathbf{j} + \mathbf{k}.$$

Thus at the point $(2, 1, 2)$:

$$\nabla w = -\frac{3}{\sqrt{8}}\mathbf{i} - \frac{2}{\sqrt{8}}\mathbf{j} + \mathbf{k},$$

and so the unit normal is

$$\frac{1}{\sqrt{21}}(- 3\mathbf{i} - 2\mathbf{j} + \sqrt{8}\mathbf{k}).$$

5. Using the fact that $\nabla\Omega$ (where $\Omega = \Omega(x, y)$ is a scalar valued function of position) is perpendicular to it's own level curves, find a unit normal to the curve defined by $x + y = 1$ at the point $(1, 0)$. Verify this result with a rough sketch.

Solution. Let $\Omega(x, y) = x + y$. Then the curve $x + y = 1$ is the level curve $\Omega = 1$. Now $\nabla\Omega = \mathbf{i} + \mathbf{j}$. So at the point $(1, 0)$ we also have $\nabla\Omega = \mathbf{i} + \mathbf{j} = (1, 1)$. The unit normal is therefore $(\mathbf{i} + \mathbf{j})/\sqrt{2}$. The sketch is shown in Figure 1.

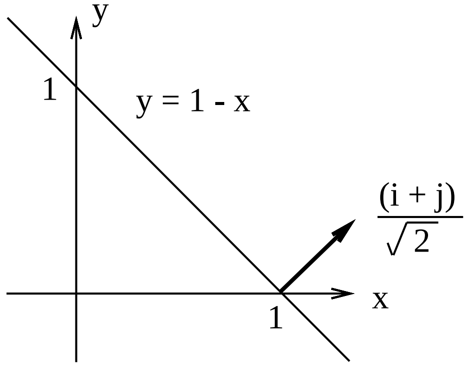

Figure 1: *Question 5.*

6. Find the derivative of the scalar field $\Omega = x^2yz + 4xz^2$ in the direction of the vector $(2, -1, -1)$ at the point $(1, -2, -1)$.

Solution. The unit vector in the direction $(2, -1, -1)$ *is*

$$\frac{(2, -1, -1)}{\sqrt{2^2 + (-1)^2 + (-1)^2}} = \frac{1}{\sqrt{6}}(2, -1, -1).$$

Now

$$\nabla\Omega = (2xyz + 4z^2, x^2z, x^2y + 8xz) \quad \Rightarrow \quad \nabla\Omega|_{(1,-2,-1)} = (8, -1, -10).$$

Thus

$$\frac{\partial\Omega}{\partial n} = (8, -1, -10) \cdot \frac{1}{\sqrt{6}}(2, -1, -1) = \frac{27}{\sqrt{6}}.$$

7. Find a unit normal to the surface $z = 3x^2y + x$ at the point $(1, 1, 4)$.

Solution. Let $\Omega = 3x^2y + x - z$. *Then* $\nabla\Omega = (6xy + 1, 3x^2, -1)$ *and so* $\nabla\Omega|_{(1,1,4)} = (7, 3, -1)$. *Hence*

$$|\nabla\Omega|_{(1,1,4)}| = \sqrt{7^2 + 3^2 + (-1)^2} = \sqrt{59}.$$

Hence the unit vector normal to the surface is $(7, 3, -1)/\sqrt{59}$.

8. Use Taylor's series in two dimensions to find a first order approximation for $f(2.5, 1)$ based on quantities evaluated at the point $(2, 1.5)$ in the case where

$$f(x, y) = x^2y + e^{xy}.$$

Solution. Here $(x_0, y_0) = (2, 1.5)$ *and (x,y) = (2.5,1). The Taylor's series is:*

$$f(x, y) = f(x_0, y_0) + \delta\mathbf{r} \cdot \nabla f|_{(x_0,y_0)} + O(|\delta\mathbf{r}|^2).$$

Now $\delta\mathbf{r} = (h, k)$, $h = x - x_0 = 2.5 - 2 = 0.5$ *and* $k = y - y_0 = 1 - 1.5 = -0.5$. *So,*

$$\delta\mathbf{r} = (0.5, -0.5) \qquad \Rightarrow \qquad |\delta\mathbf{r}| = \sqrt{0.5}.$$

Also

$$\nabla f = (2xy + ye^{xy})\mathbf{i} + (x^2 + xe^{xy})\mathbf{j} \qquad \Rightarrow \qquad \nabla f|_{(2,1.5)} = 36.128\mathbf{i} + 44.171\mathbf{j},$$

and $f(2, 1.5) = 26.086$. *So*

$$f(2.5, 1) = 26.086 + (0.5, -0.5) \cdot (36.128, 44.171) + O(0.5) = 22.0645 + O(0.5).$$

Note that the exact value is $f(2.5, 1) = 18.4325$.

9. Use Taylor's series in two dimensions to find a first order approximation of

$$f(x, y) = \sin(2xy) + e^{x+3y},$$

at the point $(1, 0)$. Hence approximate $f(1.5, 0.2)$ and estimate the size of the error.

Solution. Again Taylor's series is:

$$f(x, y) = f(x_0, y_0) + \delta\mathbf{r} \cdot \nabla f|_{(x_0,y_0)} + O(|\delta\mathbf{r}|^2).$$

Here $(x_0, y_0) = (1, 0)$ *and*

$$\delta\mathbf{r} = (h, k) = (x - x_0, y - y_0) = (x - 1, y),$$

and

$$\nabla f = (2y\cos(2xy) + e^{x+3y}, 2x\cos(2xy) + 3e^{x+3y}).$$

So

$$\nabla f|_{(1,0)} = (e^1, 2 + 3e^1) = (2.718, 10.155).$$

Also $f(1, 0) = e^1 = 2.718$. *Hence*

$$\begin{aligned} f(x, y) &= 2.718 + (x - 1, y) \cdot (2.718, 10.155) + O(|\delta\mathbf{r}|^2) \\ &= 2.718 + 2.718(x - 1) + 10.155y + O(|\delta\mathbf{r}|^2) \\ &= 2.718x + 10.155y + O(|\delta\mathbf{r}|^2). \end{aligned}$$

If $(x, y) = (1.5, 0.2)$ *then*

$$\delta\mathbf{r} = (0.5, 0.2) \qquad \Rightarrow \qquad |\delta\mathbf{r}|^2 = 0.25 + 0.04 = 0.29.$$

Hence

$$f(1.5, 0.2) = 2.718(1.5) + 10.155(0.2) + O(0.29) = 6.108 + O(0.29).$$

Note the exact value is $f(1.5, 0.2) = 8.73$.

10. If $\Omega = xy^3z^3 + x^3y^2z$, find grad Ω at the point $(1, -1, 1)$.

Solution.

$$\nabla\Omega = (y^3z^3 + 3x^2y^2z, 3xy^2z^3 + 2x^3yz, 3xy^3z^2 + x^3y^2).$$

So

$$\nabla\Omega|_{(1,-1,1)} = (2, 1, -2).$$

11. If $\Omega = x^n + y^n + z^n$ (with n a known constant) and if $\mathbf{r} = (x, y, z)$ then show that $\mathbf{r} \cdot \nabla\Omega = n\Omega$.

Solution. Now

$$\nabla\Omega = (nx^{n-1}, ny^{n-1}, nz^{n-1}).$$

So

$$\mathbf{r} \cdot \nabla\Omega = (x, y, z) \cdot (nx^{n-1}, ny^{n-1}, nz^{n-1}) = nx^n + ny^n + nz^n = n(x^n + y^n + z^n) = n\Omega.$$

12. Find the divergence and curl of $\mathbf{f} = (xy, yz, 0)$ at the point $(1, 1, 1)$. Also evaluate $\nabla(\nabla \cdot \mathbf{f})$.

Solution.

$$div\,\mathbf{f} = \nabla \cdot \mathbf{f} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (xy, yz, 0) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(0) = y + z.$$

Also

$$curl\,\mathbf{f} = \nabla \times \mathbf{f} \quad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & 0 \end{vmatrix}$$

$$
\begin{aligned}
&= \mathbf{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(yz)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(xy)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right) \\
&= (-y, 0, -x).
\end{aligned}
$$

Hence $div\,\mathbf{f}|_{(1,1,1)} = 1 + 1 = 2$ *and* $curl\,\mathbf{f}|_{(1,1,1)} = (-1, 0, -1)$. *Also*

$$
\nabla(\nabla \cdot \mathbf{f}) = \nabla(y + z) = \left(\frac{\partial}{\partial x}(y + z), \frac{\partial}{\partial y}(y + z), \frac{\partial}{\partial z}(y + z)\right) = (0, 1, 1).
$$

13. If $\Omega = x + y^2 + z^3$, find div(gradΩ) and curl(gradΩ).

Solution.

$$
div\,(grad\Omega) = \nabla \cdot (\nabla\Omega).
$$

Now $\nabla\Omega = (1, 2y, 3z^2)$ *and so*

$$
div\,(grad\Omega) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z^2) = 2 + 6z.
$$

$$
\begin{aligned}
curl\,(grad\Omega) = \nabla \times \nabla\Omega &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 2y & 3z^2 \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(3z^2) - \frac{\partial}{\partial z}(2y)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(3z^2) - \frac{\partial}{\partial z}(1)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(zy) - \frac{\partial}{\partial y}(1)\right) \\
&= (0, 0, 0).
\end{aligned}
$$