MA4006: Exercise Sheet 1: Solutions

1. Sketch the curves represented by the following parametric equations:

(i) $\mathbf{r}(t) = (2\cos\pi t, \sin\pi t, 0)$, (ii) $\mathbf{r}(t) = (\sin\pi t, 0, 0)$, (iii) $\mathbf{r}(t) = (t, |t|, 0)$

(iv) $\mathbf{r}(t) = (t, -t, 0)$, $-\infty < t \leq 0$, (v) $\mathbf{r}(t) = (t, -t^2, 0)$, $0 < t \leq \infty$.

Solution (i). $x = 2\cos\pi t$, $y = \sin\pi t$ *and* $z = 0$. *Thus*

$$\frac{x^2}{4} + y^2 = \frac{(2\cos\pi t)^2}{4} + (\sin\pi t)^2 = 1.$$

Hence the curve describes an ellipse in 2D, see Figure 1.

Solution (ii). $x = \sin\pi t$, $y = z = 0$. *Since* $y = 0$ *this is just a line between* -1 *and* $+1$ *on the x-axis (as* $\sin\pi t$ *lies between* -1 *and* $+1$*). See Figure 1.*

Solution (iii). $x = t$, $y = |t|$, $z = 0$. *Hence* $y = |x|$. *See Figure 1.*

Solution (iv). $x = t$, $y = -t$, $z = 0$. *Hence* $y = -x$. *Since* $-\infty < t \leq 0$ *we also have* $-\infty < x \leq 0$, *see Figure 1.*

Solution (v). $x = t$, $y = -t^2$, $z = 0$. *Hence* $y = -x^2$. *Since* $0 < t \leq \infty$ *we also have* $0 < x \leq \infty$, *see Figure 1.*

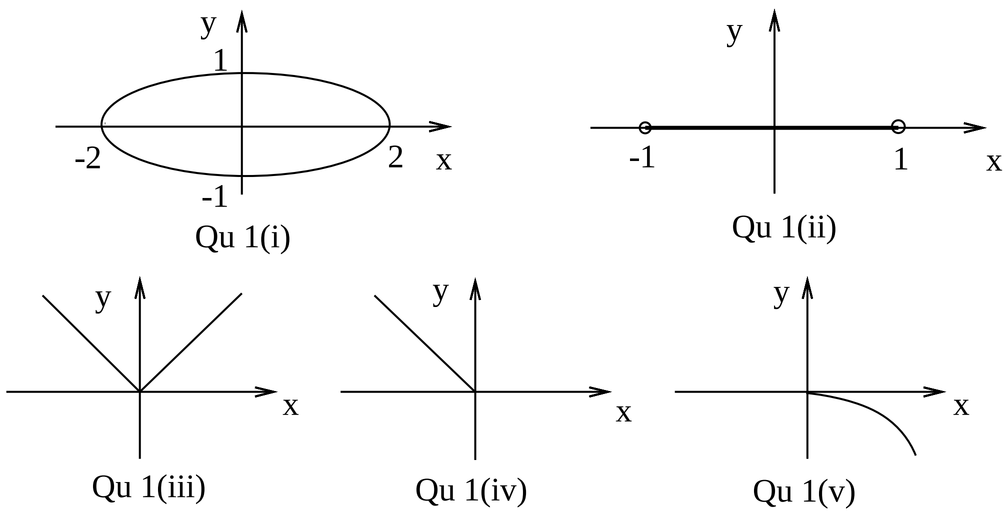

Figure 1: *Question 1.*

2. Determine the derivatives $\dfrac{d\mathbf{r}}{dt}$ and $\dfrac{d^2\mathbf{r}}{dt^2}$ of the following vectors:

(i) $\mathbf{r}(t) = (2\cos\pi t, \sin\pi t, 0)$, (ii) $\mathbf{r}(t) = (t, t, e^t)$, (iii) $\mathbf{r}(t) = (t^2, t^3 - t, 0)$.

Solution (i). $\mathbf{r}'(t) = (-2\pi \sin \pi t, \pi \cos \pi t, 0)$ *and* $\mathbf{r}''(t) = (-2\pi^2 \cos \pi t, -\pi^2 \sin \pi t, 0)$.

Solution (ii). $\mathbf{r}'(t) = (1, 1, e^t)$ *and* $\mathbf{r}''(t) = (0, 0, e^t)$.

Solution (ii). $\mathbf{r}'(t) = (2t, 3t^2 - 1, 0)$ *and* $\mathbf{r}''(t) = (2, 6t, 0)$.

3. In the following, (a) compute $\mathbf{f} \cdot \mathbf{g}$ and differentiate the resulting real-valued function and (b) compute $(\mathbf{f} \cdot \mathbf{g})'$ using the differential rule (iii) in Section 1.4 of the lecture notes.

$$\text{(i)} \qquad \mathbf{f}(t) = e^t \mathbf{j} + 2\mathbf{k}, \quad \mathbf{g}(t) = \cos t \mathbf{i} + 2\mathbf{j} + t^2 \mathbf{k}$$

$$\text{(ii)} \qquad \mathbf{f}(t) = -4 \cos t \mathbf{k}, \quad \mathbf{g}(t) = -t^2 \mathbf{i} + 4 \sin t \mathbf{k}.$$

Solution (i). Now

$$\mathbf{f} \cdot \mathbf{g} = (0, e^t, 2) \cdot (\cos t, 2, t^2) = 2e^t + 2t^2 \qquad \Rightarrow \qquad (\mathbf{f} \cdot \mathbf{g})' = 2e^t + 4t.$$

Also, the differential rule (iii) in Section 1.4 says $(\mathbf{f} \cdot \mathbf{g})' = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$. *Thus*

$$(\mathbf{f} \cdot \mathbf{g})' = (0, e^t, 0) \cdot (\cos t, 2, t^2) + (0, e^t, 2) \cdot (-\sin t, 0, 2t) = 2e^t + 4t.$$

Solution (ii). Now

$$\mathbf{f} \cdot \mathbf{g} = (0, 0, -4 \cos t) \cdot (-t^2, 0, 4 \sin t) = -16 \cos t \sin t = -8 \sin(2t),$$

and so $(\mathbf{f} \cdot \mathbf{g})' = -16 \cos(2t)$.

Also, the differential rule (iii) in Section 1.4 says $(\mathbf{f} \cdot \mathbf{g})' = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$. *Thus*

$$\begin{aligned} (\mathbf{f} \cdot \mathbf{g})' &= (0, 0, 4 \sin t) \cdot (-t^2, 0, 4 \sin t) + (0, 0, -4 \cos t) \cdot (-2t, 0, 4 \cos t) \\ &= 16 \sin^2 t - 16 \cos^2 t \\ &= -16 \cos(2t). \end{aligned}$$

4. In the following, (a) compute $\mathbf{f} \times \mathbf{g}$ and differentiate the resulting real-valued function and (b) compute $(\mathbf{f} \times \mathbf{g})'$ using the differential rule (iv) in Section 1.4 of the lecture notes.

$$\text{(i)} \qquad \mathbf{f}(t) = t\mathbf{i} + \mathbf{j} + 4\mathbf{k}, \quad \mathbf{g}(t) = \mathbf{i} - \cos t \mathbf{j} + t\mathbf{k}$$

$$\text{(ii)} \qquad \mathbf{f}(t) = -9\mathbf{i} + t^2 \mathbf{j} + t^2 \mathbf{k}, \quad \mathbf{g}(t) = e^t \mathbf{i}.$$

Solution (i). Now

$$\mathbf{f}\times\mathbf{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 4 \\ 1 & -\cos t & t \end{vmatrix} = (t+4\cos t)\mathbf{i} - (t^2-4)\mathbf{j} + (-t\cos t - 1)\mathbf{k}.$$

Hence

$$(\mathbf{f}\times\mathbf{g})' = (1-4\sin t)\mathbf{i} - 2t\mathbf{j} + (-\cos t + t\sin t)\mathbf{k}.$$

Also, the differential rule (iv) in Section 1.4 says $(\mathbf{f}\times\mathbf{g})' = (\mathbf{f}'\times\mathbf{g}) + (\mathbf{f}\times\mathbf{g}')$. *Thus*

$$\mathbf{f}'\times\mathbf{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & -\cos t & t \end{vmatrix} = -t\mathbf{j} - \cos t\mathbf{k}$$

$$\mathbf{f}\times\mathbf{g}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 4 \\ 0 & \sin t & 1 \end{vmatrix} = (1-4\sin t)\mathbf{i} - t\mathbf{j} + t\sin t\mathbf{k},$$

and so

$$(\mathbf{f}\times\mathbf{g})' = (1-4\sin t)\mathbf{i} - 2t\mathbf{j} + (-\cos t + t\sin t)\mathbf{k}.$$

Solution (ii). Now

$$\mathbf{f}\times\mathbf{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -9 & t^2 & t^2 \\ e^t & 0 & 0 \end{vmatrix} = t^2e^t\mathbf{j} - t^2e^t\mathbf{k}.$$

Hence

$$(\mathbf{f}\times\mathbf{g})' = (2te^t + t^2e^t)\mathbf{j} - (2te^t + t^2e^t)\mathbf{k}.$$

Also, the differential rule (iv) in Section 1.4 says $(\mathbf{f}\times\mathbf{g})' = (\mathbf{f}'\times\mathbf{g}) + (\mathbf{f}\times\mathbf{g}')$. *Thus*

$$\mathbf{f}'\times\mathbf{g} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2t & 2t \\ e^t & 0 & 0 \end{vmatrix} = 2te^t\mathbf{j} - 2te^t\mathbf{k}$$

$$\mathbf{f}\times\mathbf{g}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -9 & t^2 & t^2 \\ e^t & 0 & 0 \end{vmatrix} = t^2e^t\mathbf{j} - t^2e^t\mathbf{k},$$

and so

$$(\mathbf{f}\times\mathbf{g})' = (2te^t + t^2e^t)\mathbf{j} - (2te^t + t^2e^t)\mathbf{k}.$$

5. Find the unit tangent vector to the curve $\mathbf{r} = (3, t, t^2)$.

Solution. Now $\mathbf{r}'(t) = (0, 1, 2t)$ *and* $|\mathbf{r}'(t)| = \sqrt{1+4t^2}$ *and so*

$$\hat{\mathbf{t}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{(0,1,2t)}{\sqrt{1+4t^2}}.$$

6. Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (t, -t^3, 0), & \text{if } -1 \le t \le 1 \\ (t, 2-3t, 0), & \text{if } 1 < t \le 2. \end{cases}$$

Find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness.

Solution.

$$\mathbf{r}'(t) = \begin{cases} (1, -3t^2, 0), & \textit{if } -1 \le t \le 1 \\ (1, -3, 0), & \textit{if } 1 < t \le 2. \end{cases}$$

Thus

$$|\mathbf{r}'(t)| = \begin{cases} \sqrt{1+9t^4}, & \textit{if } -1 \le t \le 1 \\ \sqrt{10}, & \textit{if } 1 < t \le 2. \end{cases}$$

and so

$$\hat{\mathbf{t}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{cases} \frac{(1,-3t^2,0)}{\sqrt{1+9t^4}}, & \textit{if } -1 \le t \le 1 \\ \frac{(1,-3,0)}{\sqrt{10}}, & \textit{if } 1 < t \le 2. \end{cases}$$

As $t \to 1^+$, $\mathbf{r}'(t) = (1, -3, 0)$ *and as* $t \to 1^-$, $\mathbf{r}'(t) = (1, -3, 0)$. *Hence the curve is smooth. Also* $\mathbf{r}'(t)$ *exists at all points and is continuous.*

7. Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (t^2, t, 0), & \text{if } -1 \le t \le 1 \\ (1, t, 0), & \text{if } 1 < t \le 2. \end{cases}$$

Find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness.

Solution.

$$\mathbf{r}'(t) = \begin{cases} (2t, 1, 0), & \text{if } -1 \le t \le 1 \\ (0, 1, 0), & \text{if } 1 < t \le 2. \end{cases}$$

Thus

$$|\mathbf{r}'(t)| = \begin{cases} \sqrt{1+4t^2}, & \text{if } -1 \le t \le 1 \\ 1, & \text{if } 1 < t \le 2. \end{cases}$$

and so

$$\hat{\mathbf{t}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{cases} \frac{(2t,1,0)}{\sqrt{1+4t^2}}, & \text{if } -1 \le t \le 1 \\ (0, 1, 0), & \text{if } 1 < t \le 2. \end{cases}$$

As $t \to 1^+$, $\mathbf{r}'(t) = (2, 1, 0)$ *but as* $t \to 1^-$, $\mathbf{r}'(t) = (0, 1, 0)$. *Now* $\mathbf{r}'(t)$ *exists at all points and is continuous, but* $\mathbf{r}'(t)$ *has a jump in value at* $t = 1$. *Hence it is not smooth (only piecewise smooth).*

8. Find the intrinsic equation of the curve $\mathbf{r} = (a\cos t, a\sin t, bt)$, $(0 \le t \le 3\pi)$.

Solution. $\mathbf{r}'(t) = (-a\sin t, a\cos t, b)$ *and so*

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2}.$$

Hence

$$s = \int_0^t |\mathbf{r}'(t_0)\, dt_0 = \int_0^t \sqrt{a^2+b^2}\, dt_0 = \sqrt{a^2+b^2}\, t,$$

which gives

$$t = \frac{s}{\sqrt{a^2+b^2}}.$$

The intrinsic equation is therefore

$$\mathbf{r}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}} \right).$$

9. Find an expression for the arclength of the curve expressed perimetrically as $\mathbf{r}(t) = (0, 5\cos t, 5\sin t)$ where $0 \le t \le 2\pi$. Using the arclength find the intrinsic equation of the curve represented by $\mathbf{r}(t)$. Find an expression for the curvature of this curve at any point.

Solution. $\mathbf{r}(t) = (0, -5\sin t, 5\cos t)$ *and so*

$$|\mathbf{r}(t)| = \sqrt{0^2 + (-5\sin t)^2 + (5\cos t)^2} = 5.$$

Hence

$$s = \int_0^t |\mathbf{r}'(t_0)\, dt_0 = \int_0^t 5\, dt_0 = 5t,$$

which gives

$$t = \frac{s}{5}.$$

The intrinsic equation is therefore

$$\mathbf{r}(s) = \left(0, 5\cos\frac{s}{5}, 5\sin\frac{s}{5}\right).$$

The curvature $\kappa(s)$ is given by

$$\kappa(s) = |\mathbf{r}''(s)|,$$

and so

$$\begin{aligned} \mathbf{r}'(s) &= \left(0, -\sin\frac{s}{5}, \cos\frac{s}{5}\right) \\ \mathbf{r}''(s) &= \left(0, -\frac{1}{5}\cos\frac{s}{5}, -\frac{1}{5}\sin\frac{s}{5}\right) \\ \Rightarrow \quad \kappa(s) &= \sqrt{\left(-\frac{1}{5}\cos\frac{s}{5}\right)^2 + \left(-\frac{1}{5}\sin\frac{s}{5}\right)^2} = \frac{1}{5}. \end{aligned}$$

10. Sketch the curve $\mathbf{r}(t) = (t, t+1)$ $0 \le t \le 1$ in two dimensions and find an expression for its arclength. Hence find the arclength from $t = 0$ to $t = 1$. Write the equation for the curve in intrinsic form.

Solution. $x = t$ and $y = t + 1$ which gives $y = x + 1$. Since $0 \le t \le 1$ we have $0 \le x \le 1$. The curve is plotted in Figure 2: Now $\mathbf{r}'(t) = (1, 1)$ and so $|\mathbf{r}'(t)| = \sqrt{2}$. Therefore

$$s(t) = \int_0^t \sqrt{2}\, dt_0 = \sqrt{2}\, t.$$

The arclength from $t = 0$ to $t = 1$ is thus $s(1) = \sqrt{2}$. Also, $t = s/\sqrt{2}$ and so the intrinsic equation is

$$\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} + 1\right).$$

11. Sketch the curve $\mathbf{r}(t) = 4t\mathbf{i} + t^2\mathbf{j}$ in two dimensions and find the velocity, speed and acceleration at time t. Hence sketch the velocity and acceleration vectors when $t = -1$.

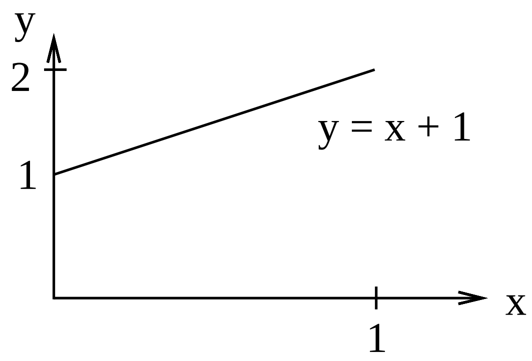

Figure 2: *Question 10.*

Solution. $x = 4t$, $y = t^2$ *and so* $t = x/4$ *giving*

$$y = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}.$$

A sketch is shown in Figure 3. Now $\mathbf{v} = \mathbf{r}'(t) = (4, 2t)$ *and so the speed is* $|\mathbf{v}| = \sqrt{4^2 + (2t)^2} = \sqrt{16 + 4t^2}$. *Also* $\mathbf{a} = \mathbf{r}''(t) = \mathbf{v}'(t) = (0, 2)$. *So* $\mathbf{v}(-1) = (4, -2)$ *and* $\mathbf{a}(-1) = (0, 2)$.

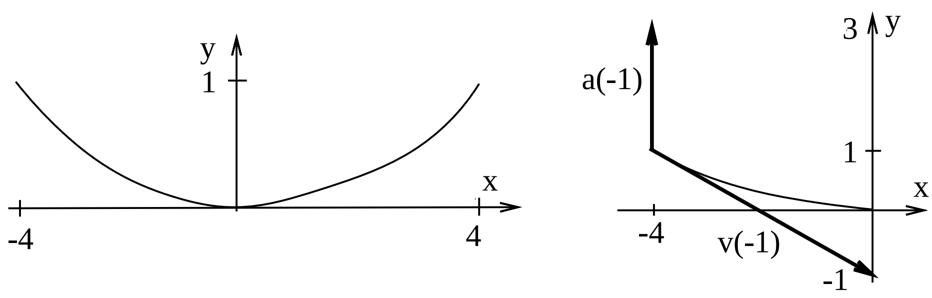

Figure 3: *Question 10.*