1. The augmented matrix is

$$\begin{pmatrix} 4 & -2 & 5 & \vdots & 1 \\ 2 & 6 & -3 & \vdots & -1 \\ 1 & 7 & -2 & \vdots & -4 \end{pmatrix}$$

The forward elimination (with "multipliers" written in the appropriate subdiagonal locations)

$$\begin{pmatrix} 4 & -2 & 5 & \vdots & 1 \\ \boxed{\frac{1}{2}} & 7 & -\frac{11}{2} & \vdots & -\frac{3}{2} \\ \boxed{\frac{1}{4}} & \frac{15}{2} & -\frac{13}{4} & \vdots & -\frac{17}{4} \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & 5 & \vdots & 1 \\ \boxed{\frac{1}{2}} & 7 & -\frac{11}{2} & \vdots & -\frac{3}{2} \\ \boxed{\frac{1}{4}} & \boxed{\frac{15}{14}} & \frac{37}{14} & \vdots & -\frac{37}{14} \end{pmatrix}$$

The backward substitution

$$\begin{aligned} 4x - 2y + 5z &= 1 \\ 7y - \frac{11}{2}z &= -\frac{3}{2} \\ \frac{37}{14}z &= -\frac{37}{14} \end{aligned}$$

yields

$$\begin{aligned} z &= -1 \\ y &= \left(-\frac{3}{2} + \frac{11}{2} \times (-1)\right)/7 = -1 \\ x &= (1 + 2 \times (-1) - 5 \times (-1))/4 = 1 \end{aligned}$$

2. Computing the factorisation using forward elimination part of *Gaussian* elimination algorithm (with "multipliers" written in the appropriate subdiagonal locations)

$$\begin{pmatrix} 2 & 4 & 4 \\ 1 & 0 & 3 \\ 2 & -7 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 4 \\ \boxed{\frac{1}{2}} & -2 & 1 \\ \boxed{1} & -11 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 4 \\ \boxed{\frac{1}{2}} & -2 & 1 \\ \boxed{1} & \boxed{\frac{11}{2}} & -\frac{9}{2} \end{pmatrix}$$

Hence

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{11}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & -\frac{9}{2} \end{pmatrix}$$

(Check that $LU = A$).
Alternatively, by *Doolittle*'s method

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

Equating the right hand side to A gives

$$\begin{aligned} u_{11} &= 2 \\ l_{21}u_{11} &= 1 \Rightarrow l_{21} = \frac{1}{2} \\ l_{31}u_{11} &= 2 \Rightarrow l_{31} = 1 \\ u_{12} &= 4 \\ l_{21}u_{12} + u_{22} &= 0 \Rightarrow u_{22} = -2 \\ l_{31}u_{12} + l_{32}u_{22} &= -7 \Rightarrow l_{32} = \frac{11}{2} \\ u_{13} &= 4 \\ l_{21}u_{13} + u_{23} &= 3 \Rightarrow u_{23} = 1 \\ l_{31}u_{13} + l_{32}u_{23} + u_{33} &= 5 \Rightarrow u_{33} = -\frac{9}{2} \end{aligned}$$

resulting in the same L and U matrices as before.

- *Solving* $Ax_1 = b_1$.
  Solving $Ly_1 = b_1$ using forward substitution:

$$\begin{aligned} y_{11} &= 2 \\ \frac{1}{2}y_{11} + y_{12} &= -4 \\ y_{11} + \frac{11}{2}y_{12} + y_{13} &= -21 \end{aligned}$$

  yields

$$\begin{aligned} y_{11} &= 2 \\ y_{12} &= -4 - \frac{1}{2} \times 2 = -5 \\ y_{13} &= -21 - 2 - \frac{11}{2} \times (-5) = \frac{9}{2} \end{aligned}$$

  and then solving $Ux_1 = y_1$ using backward substitution:

$$\begin{aligned} 2x_{11} + 4x_{12} + 4x_{13} &= 2 \\ -2x_{12} + x_{13} &= -5 \\ -\frac{9}{2}x_{13} &= \frac{9}{2} \end{aligned}$$

  yields

$$\begin{aligned} x_{13} &= -1 \\ x_{12} &= (-5 + 1)/(-2) = 2 \\ x_{11} &= (2 - 4 \times (-1) - 4 \times (2))/2 = -1 \end{aligned}$$

- *Solving* $Ax_2 = x_1$.
  Solving $Ly_2 = x_1$ using forward substitution:

$$\begin{aligned} y_{21} &= -1 \\ \frac{1}{2}y_{21} + y_{22} &= 2 \\ y_{21} + \frac{11}{2}y_{22} + y_{23} &= -1 \end{aligned}$$

yields

$$
\begin{aligned}
y_{21} &= -1 \\
y_{22} &= 2 - \frac{1}{2} \times (-1) = \frac{5}{2} \\
y_{23} &= -1 + 1 - \frac{11}{2} \times (\frac{5}{2}) = -\frac{55}{4}
\end{aligned}
$$

and then solving $Ux_2 = y_2$ using backward substitution:

$$
\begin{aligned}
2x_{21} + 4x_{22} + 4x_{23} &= -1 \\
-2x_{22} + x_{23} &= \frac{5}{2} \\
-\frac{9}{2}x_{13} &= -\frac{55}{4}
\end{aligned}
$$

yields

$$
\begin{aligned}
x_{23} &= \frac{55}{18} \\
x_{22} &= \left(\frac{5}{2} - \frac{55}{18}\right)/(-2) = \frac{5}{18} \\
x_{21} &= \left(-1 - 4 \times (\frac{55}{18}) - 4 \times (\frac{5}{18})\right)/2 = -\frac{43}{6}
\end{aligned}
$$

3. Using *Doolittle*'s method

$$
LL^T = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix} = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}
$$

Equating the right hand side to A gives (notice how the symmetry reduces the number of independent equations)

$$
\begin{aligned}
l_{11}^2 &= 2 \Rightarrow l_{11} = \sqrt{2} \\
l_{21}l_{11} &= 0 \Rightarrow l_{21} = 0 \\
l_{31}l_{11} &= 1 \Rightarrow l_{31} = \frac{1}{\sqrt{2}} \\
l_{21}^2 + l_{22}^2 &= 4 \Rightarrow l_{22} = 2 \\
l_{31}l_{21} + l_{32}l_{22} &= -2 \Rightarrow l_{32} = -1 \\
l_{31}^2 + l_{32}^2 + l_{33}^2 &= 3 \Rightarrow l_{33} = \frac{\sqrt{3}}{\sqrt{2}}
\end{aligned}
$$

Hence

$$
L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix}
$$

Solving $Ax = b$.
Solving $Ly = b$ using forward substitution

$$
\begin{aligned}
\sqrt{2}y_1 &= 9 \\
2y_2 &= 6 \\
\frac{1}{\sqrt{2}}y_1 - y_2 + \frac{\sqrt{3}}{\sqrt{2}}y_3 &= 0
\end{aligned}
$$

yields

$$\begin{aligned} y_1 &= \frac{9}{\sqrt{2}} \\ y_2 &= 3 \\ y_3 &= \left(-\frac{1}{\sqrt{2}} \times \frac{9}{\sqrt{2}} + 3\right) \times \frac{\sqrt{2}}{\sqrt{3}} = -\frac{\sqrt{3}}{\sqrt{2}} \end{aligned}$$

and then solving $L^T x = y$ using backward substitution

$$\begin{aligned} \sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_3 &= \frac{9}{\sqrt{2}} \\ 2x_2 - x_3 &= 3 \\ \frac{\sqrt{3}}{\sqrt{2}}x_3 &= -\frac{\sqrt{3}}{\sqrt{2}} \end{aligned}$$

gives

$$\begin{aligned} x_3 &= -1 \\ x_2 &= (3-1)/2 = 1 \\ x_1 &= \left(\frac{9}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \times \frac{1}{\sqrt{2}} = 5 \end{aligned}$$

4. (Recall that in the context of this problem, we are using the "max" norm for vectors and "maximum row sum" norm for matrices).

   (a) $||A|| = \max(0.21, 4.1) = 4.1$. For the given 2×2 matrix, it is straightforward to compute

   $$A^{-1} = \begin{pmatrix} -5 & 0.5 \\ 200 & -10 \end{pmatrix}$$

   and hence $||A^{-1}|| = \max(5.05, 210) = 210$. Thus $\kappa(A) = 861$.

   (b) $||A|| = \max(10, 4, 14) = 14$. What is $||A^{-1}||$?
   In general it is too expensive to compute A^{-1}, and so it is necessary to estimate $||A^{-1}||$. We do this as follows: For any list of vectors b_i, $i = 1, 2, \ldots m$, let the list x_i represent the solutions of $Ax_i = b_i$. (If the LU factorisation of A is known, then it is straightforward to solve these systems.) Then since $x_i = A^{-1}b_i$ we have $||x_i|| \leq ||A^{-1}||\ ||b_i||$ or $||A^{-1}|| \geq ||x_i||/||b_i||$ and hence that

   $$||A^{-1}|| \approx \max_{1 \leq i \leq m} \frac{||x_i||}{||b_i||}$$

   For $b_1 = (1, 0, 0)^T$, we get $x_1 = (21/18,\ 1/18,\ -7/18)^T$. Thus $||b_1|| = 1$ and $||x_1|| = 21/18$.
   For $b_2 = (0, 1, 0)^T$, we obtain $x_2 = (-48/18,\ 2/18,\ 22/18)^T$. Thus $||b_2|| = 1$ while $||x_2|| = 48/18$. From this short list we would estimate

   $$||A^{-1}|| \approx \max(7/6, 8/3) = 8/3$$

   and hence $\kappa(A) \approx 37.33$.
   Addendum: For those intent on improving their *Gauss - Jordan* technique, it may be shown that

   $$A^{-1} = \frac{1}{18}\begin{pmatrix} 21 & -48 & 12 \\ 1 & 2 & -2 \\ -7 & 22 & -4 \end{pmatrix}$$

   which results in $||A^{-1}|| = 9/2$ and $\kappa(A) = 63$.

(c) We proceed as per part (b). $||A|| = 6$.
For $b_1 = (1,0,0)^T$, we get $x_1 = (2/3,\ -1/6,\ -1/3)^T$. Thus $||b_1|| = 1$ and $||x_1|| = 2/3$.
For $b_2 = (0,1,0)^T$, we obtain $x_2 = (-1/6,\ 5/12,\ 1/3)^T$. Thus $||b_2|| = 1$ while $||x_2|| = 5/12$. From this short list we would estimate

$$||A^{-1}|| \approx \max(2/3, 5/12) = 2/3$$

and hence $\kappa(A) \approx 4$.

5. (a) For the *Jacobi* method, the iteration matrix $G = I - D^{-1}A$, where D is the diagonal matrix with entries equal to the diagonal entries of A (see Notes). In this example $D = I$ and so $G = I - A$. We compute that $det(G - \lambda I) = -\lambda^3 + (144/169)\lambda$, and thus $\rho(G) = \max|\lambda_i| = 12/13 < 1$ which guarantees convergence of the method.

(b) The *Jacobi* method uses the iteration: $x(k+1) = Gx(k) + D^{-1}b$. In this example, this becomes

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 8/13 \\ -8/13 & 0 & -8/13 \\ -8/13 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + \begin{pmatrix} -22 \\ 12 \\ 9 \end{pmatrix}$$

and so we get the sequence

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -22 \\ 12 \\ 9 \end{pmatrix} \mapsto \begin{pmatrix} -28.461 \\ 20.000 \\ 10.538 \end{pmatrix} \mapsto \begin{pmatrix} -35.515 \\ 23.030 \\ 6.515 \end{pmatrix} \mapsto \begin{pmatrix} -41.020 \\ 29.846 \\ 7.826 \end{pmatrix}$$

(c) For the *Gauss-Seidel* method, the iteration matrix $G = I - L^{-1}A$ where L is the "triangular part" of A (see Notes). In this example

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 8/13 & 1 & 0 \\ 8/13 & 1 & 1 \end{pmatrix} \Rightarrow L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -8/13 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow G = \begin{pmatrix} 0 & -1 & 8/13 \\ 0 & 8/13 & -(8/13 + 64/169) \\ 0 & 0 & 8/13 \end{pmatrix}$$

We can compute that $det(G - \lambda I) = -\lambda\,(\lambda - (8/13))^2$, and thus $\rho(G) = \max|\lambda_i| = 8/13 < 1$ which guarantees convergence of the method.