1. • *Matrix* $A_1$

$$\det(\lambda I - A_1) = \begin{vmatrix} \lambda - 2 & -1 \\ -3 & \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

Thus the characteristic equation $\det(\lambda I - A_1) = 0$ yields

$$\lambda_1 = -1, \qquad \lambda_2 = 3$$

(a) For $\lambda_1 = -1$, we have that $(\lambda I - A_1)e = 0$ becomes

$$\begin{pmatrix} -3 & -1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{12} = -3e_{11}$$

Hence the eigenspace corresponding to $\lambda_1 = -1$ is

$$E_{-1} = \left\{ \alpha \begin{pmatrix} 1 \\ -3 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

(b) For $\lambda_2 = 3$, we have that $(\lambda I - A_1)e = 0$ becomes

$$\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{22} = e_{21}$$

Hence the eigenspace corresponding to $\lambda_2 = 3$ is

$$E_3 = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

• *Matrix* $A_2$

$$\det(\lambda I - A_2) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & -1 & \lambda - 1 \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 = (\lambda - 1)^2(\lambda + 1)$$

Thus the characteristic equation $\det(\lambda I - A_2) = 0$ yields

$$\lambda_1 = -1, \qquad \lambda_2 = 1(\text{multiplicity} = 2)$$

(a) For $\lambda_1 = -1$, we have that $(\lambda I - A_2)e = 0$ becomes

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{12} = -e_{11}, \quad e_{13} = e_{11}$$

Hence the eigenspace corresponding to $\lambda_1 = -1$ is

$$E_{-1} = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

(b) For $\lambda_2 = 1$, we have that $(\lambda I - A_2)e = 0$ becomes

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{22} = e_{21}, \quad e_{23} = e_{21}$$

Hence the eigenspace corresponding to $\lambda_2 = 1$ is

$$E_1 = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

- *Matrix* $A_3$

$$\det(\lambda I - A_3) = \begin{vmatrix} \lambda-1 & 3 & 1 \\ -1 & \lambda & 1 \\ 3 & 3 & \lambda-2 \end{vmatrix} = \lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda+1)(\lambda-1)(\lambda-3)$$

Thus the characteristic equation $\det(\lambda I - A_3) = 0$ yields

$$\lambda_1 = -1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 3$$

(a) For $\lambda_1 = -1$, we have that $(\lambda I - A_3)e = 0$ becomes

$$\begin{pmatrix} -2 & 3 & 1 \\ -1 & -1 & 1 \\ 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{11} = 4e_{12}, \quad e_{13} = 5e_{12}$$

Hence the eigenspace corresponding to $\lambda_1 = -1$ is

$$E_{-1} = \left\{ \alpha \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

(b) For $\lambda_2 = 1$, we have that $(\lambda I - A_3)e = 0$ becomes

$$\begin{pmatrix} 0 & 3 & 1 \\ -1 & 1 & 1 \\ 3 & 3 & -1 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{21} = -2e_{22}, \quad e_{23} = -3e_{22}$$

Hence the eigenspace corresponding to $\lambda_2 = 1$ is

$$E_1 = \left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

(c) For $\lambda_3 = 3$, we have that $(\lambda I - A_3)e = 0$ becomes

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{31} = 0, \quad e_{33} = -3e_{32}$$

Hence the eigenspace corresponding to $\lambda_3 = 3$ is

$$E_3 = \left\{ \alpha \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

- *Matrix* $A_4$. Note the matrix is symmetric.

$$\det(\lambda I - A_4) = \begin{vmatrix} \lambda & -2 & 2 \\ -2 & \lambda+3 & -4 \\ 2 & -4 & \lambda+5 \end{vmatrix} = \lambda^3 + 8\lambda^2 - 9\lambda = \lambda(\lambda+9)(\lambda-1)$$

Thus the characteristic equation $\det(\lambda I - A_4) = 0$ yields

$$\lambda_1 = -9, \qquad \lambda_2 = 0, \qquad \lambda_3 = 1$$

(a) For $\lambda_1 = -9$, we have that $(\lambda I - A_4)e = 0$ becomes

$$\begin{pmatrix} -9 & -2 & 2 \\ -2 & -6 & -4 \\ 2 & -4 & -4 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{21} = -2e_{11}, \quad e_{13} = \frac{5}{2}e_{11}$$

Hence the eigenspace corresponding to $\lambda_1 = -9$ is

$$E_{-9} = \left\{ \alpha \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

(b) For $\lambda_2 = 0$, we have that $(\lambda I - A_4)e = 0$ becomes

$$\begin{pmatrix} 0 & -2 & 2 \\ -2 & 3 & -4 \\ 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{22} = -2e_{21}, \quad e_{23} = -2e_{21}$$

Hence the eigenspace corresponding to $\lambda_2 = 0$ is

$$E_0 = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

(c) For $\lambda_3 = 1$, we have that $(\lambda I - A_4)e = 0$ becomes

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 6 \end{pmatrix} \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad e_{31} = 2e_{32}, \quad e_{33} = 0$$

Hence the eigenspace corresponding to $\lambda_3 = 1$ is

$$E_1 = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$$

Note that

$$\left\{ \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is an orthogonal set (using the standard inner product on $\mathbf{R}^3$).

2. All except $A_2$ are diagonalisable: $A_1$, $A_3$ and $A_4$ have $n$ distinct eigenvalues, while $A_2$ does not have $n = 3$ linearly independent eigenvectors.
Let $P_i$ be a diagonalising matrix for $A_i$ ( chosen using the eigenvalue ordering of Question (1)). Then we have

$$\begin{aligned} P_1 &= \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \\ P_3 &= \begin{pmatrix} 4 & -2 & 0 \\ 1 & 1 & 1 \\ 5 & 3 & -3 \end{pmatrix} \\ P_4 &= \begin{pmatrix} 2 & 1 & 2 \\ -4 & -2 & 1 \\ 5 & -2 & 0 \end{pmatrix} \end{aligned}$$

Note that even though the columns of $P_4$ are orthogonal, they are not orthonormal and thus $P_4$ is not orthogonal.

3. $A_4$, being symmetric, is orthogonally diagonalisable. We need to choose orthonormal eigenvectors to make $P$ orthogonal. This is possible since we can take any (non-zero) multiple of a basic eigenvector.
Thus normalising the columns of $P_4$ of Question (2) gives

$$P = \begin{pmatrix} \frac{2}{\sqrt{45}} & \frac{1}{3} & \frac{2}{\sqrt{5}} \\ -\frac{4}{\sqrt{45}} & -\frac{2}{3} & \frac{1}{\sqrt{5}} \\ \frac{5}{\sqrt{45}} & -\frac{2}{3} & 0 \end{pmatrix}$$

We can show that $P^{-1} = P^T$ by showing that $P^T P = I$.

4. For a diagonalisable matrix, we know that

$$A^k = PD^kP^{-1}$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $P = [e_1, e_2, \cdots e_n]$. Since both $A_1$ and $A_4$ are diagonalisable, we get

$$\begin{aligned} A_1^5 &= \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} (-1)^5 & 0 \\ 0 & 3^5 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{(-1)^5+3^6}{4} & \frac{(-1)^6+3^5}{4} \\ \frac{-3(-1)^5+3^6}{4} & \frac{3(-1)^5+3^5}{4} \end{pmatrix} = \begin{pmatrix} 182 & 61 \\ 183 & 60 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A_4^4 &= \begin{pmatrix} \frac{2}{\sqrt{45}} & \frac{1}{3} & \frac{2}{\sqrt{5}} \\ -\frac{4}{\sqrt{45}} & -\frac{2}{3} & \frac{1}{\sqrt{5}} \\ \frac{5}{\sqrt{45}} & -\frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} (-9)^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{4}{\sqrt{45}} & \frac{5}{\sqrt{45}} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 584 & -1166 & 1458 \\ -1166 & 2333 & -2916 \\ 1458 & -2916 & 3645 \end{pmatrix} \end{aligned}$$

5. (a) Letting $\dot{x}$ stand for $\frac{dx}{dt}$ etc., gives

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 \\ 2 & -3 & 4 \\ -2 & 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

Note that the system matrix is $A_4$ of Questions (1) and (3).

(b) Using the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{45}} & \frac{1}{3} & \frac{2}{\sqrt{5}} \\ -\frac{4}{\sqrt{45}} & -\frac{2}{3} & \frac{1}{\sqrt{5}} \\ \frac{5}{\sqrt{45}} & -\frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

transforms the system (as per the diagonalisation discussed) to

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{y}} \\ \dot{\hat{z}} \end{pmatrix} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}, \qquad \begin{pmatrix} \hat{x}(0) \\ \hat{y}(0) \\ \hat{z}(0) \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{45}} & -\frac{4}{\sqrt{45}} & \frac{5}{\sqrt{45}} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{45}} \\ \frac{4}{3} \\ \frac{4}{\sqrt{5}} \end{pmatrix}$$

(c) Noting that the equation $\dot{w} = kw,\ w(0) = w_0$ has solution $w(t) = e^{kt}w_0$ enables us to say that the solution for the transformed variables is

$$\begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \\ \hat{z}(t) \end{pmatrix} = \begin{pmatrix} e^{-9t}\hat{x}(0) \\ \hat{y}(0) \\ e^{t}\hat{z}(0) \end{pmatrix} = \begin{pmatrix} -e^{-9t}\frac{1}{\sqrt{45}} \\ \frac{4}{3} \\ e^{t}\frac{4}{\sqrt{5}} \end{pmatrix}$$

and so the solution of the original system is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{45}} & \frac{1}{3} & \frac{2}{\sqrt{5}} \\ -\frac{4}{\sqrt{45}} & -\frac{2}{3} & \frac{1}{\sqrt{5}} \\ \frac{5}{\sqrt{45}} & -\frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \\ \hat{z}(t) \end{pmatrix} = \begin{pmatrix} -e^{-9t}\frac{2}{45} + \frac{4}{9} + e^{t}\frac{8}{5} \\ e^{-9t}\frac{4}{45} - \frac{8}{9} + e^{t}\frac{4}{5} \\ -e^{-9t}\frac{5}{45} - \frac{8}{9} \end{pmatrix}$$