1.

$$\begin{aligned} < A, B > &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} \\ &= b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22} \\ &= < B, A > \end{aligned}$$

$$\begin{aligned} < A + C, B > &= (a_{11} + c_{11})b_{11} + (a_{12} + c_{12})b_{12} + (a_{21} + c_{21})b_{21} + (a_{22} + c_{22})b_{22} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} + c_{11}b_{11} + c_{12}b_{12} + c_{21}b_{21} + c_{22}b_{22} \\ &= < A, B > + < C, B > \end{aligned}$$

$$\begin{aligned} < \alpha A, B > &= (\alpha a_{11})b_{11} + (\alpha a_{12})b_{12} + (\alpha a_{21})b_{21} + (\alpha a_{22})b_{22} \\ &= \alpha(a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}) \\ &= \alpha < A, B > \end{aligned}$$

$$\begin{aligned} < A, A > &= a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \\ &\geq 0 \\ &= 0 \text{ only if } a_{ij} = 0 \Rightarrow A = 0 \end{aligned}$$

Note

$$||A|| = \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}$$

For the given A and B matrices

$$\begin{aligned} < A, B > &= 1 \times (-2) + 0 \times 3 + 3 \times 2 + 1 \times 4 = 8 \\ < A, A > &= 1^2 + 0^2 + 3^2 + 1^2 = 11 \\ < B, B > &= 33 \end{aligned}$$

Hence

(a) *Cauchy - Schwarz*: $8^2 \leq 11 \times 33$

(b) $||A|| = \sqrt{11}$

(c)

$$\cos\theta = \frac{< A, B >}{||A|| \times ||B||} = \frac{8}{\sqrt{11}\sqrt{33}} \Rightarrow \theta = \arccos\frac{8\sqrt{3}}{33}$$

(d) $dim\ V = 4$. An orthonormal basis is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

2. From the definition of the "max row sum " norm, it is obvious that $||A|| \geq 0$, and if $||A|| = 0$ then $a_{ij} = 0 \Rightarrow A = 0$. Similarly $||\alpha\, A|| = |\alpha|\ ||A||$ follows from the definition, and finally, since for any real numbers a_{ij}, b_{ij} we have

$$
\begin{aligned}
|a_{ij}+b_{ij}| &\leq |a_{ij}|+|b_{ij}| \\
\Rightarrow \sum_j |a_{ij}+b_{ij}| &\leq \sum_j |a_{ij}|+\sum_j |b_{ij}| \\
\Rightarrow \max_i \sum_j |a_{ij}+b_{ij}| &\leq \max_i \sum_j |a_{ij}|+\max_i \sum_j |b_{ij}|
\end{aligned}
$$

it follows that

$$
\begin{aligned}
||A+B|| &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}+b_{ij}| \\
&\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| + \max_{1\leq i\leq n} \sum_{j=1}^{n} |b_{ij}| \\
&= ||A||+||B||
\end{aligned}
$$

3. (a) Step 1. $||v_1|| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$.

$$
u_1 = \frac{v_1}{||v_1||} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}
$$

Then $W_1 = \text{span }\{u_1\}$.

Step 2. $v_2 = w_1 + w_1^{\perp}$ where $w_1 =< v_2, u_1 > u_1 \in W_1$ and $w_1^{\perp} = v_2 - w_1$ is in the orthogonal complement of W_1.
Here $< v_2, u_1 >= 0$ and so $w_1^{\perp} = v_2$. Its norm is $||w_1^{\perp}|| = \sqrt{2}$ and

$$
u_2 = \frac{w_1^{\perp}}{||w_1^{\perp}||} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}
$$

Then $W_2 = \text{span }\{u_1,\ u_2\}$.

Step 3. $v_3 = w_2 + w_2^{\perp}$ where $w_2 =< v_3, u_1 > u_1 + < v_3, u_2 > u_2 \in W_2$ and $w_2^{\perp} = v_3 - w_2$ is in the orthogonal complement of W_2.
Here $< v_3, u_1 >= 3/\sqrt{2}$, $< v_3, u_2 >= -1/\sqrt{2}$ and so

$$
w_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \Rightarrow w_2^{\perp} = v_3 - w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Its norm is $||w_2^{\perp}|| = 1$ and

$$
u_3 = \frac{w_2^{\perp}}{||w_2^{\perp}||} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

The orthonormal basis is therefore

$$
\{u_1,\ u_2,\ u_3\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
$$

(b) Step 1. $||v_1|| = \sqrt{1^2 + 2 \times 1^2 + 3 \times 0^2} = \sqrt{3}$.

$$u_1 = \frac{v_1}{||v_1||} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$$

Then $W_1 = \text{span}\,\{u_1\}$.

Step 2. $v_2 = w_1 + w_1^{\perp}$ where $w_1 =< v_2, u_1 > u_1 \in W_1$ and $w_1^{\perp} = v_2 - w_1$ is in the orthogonal complement of W_1.
Here $< v_2, u_1 >= 1 \times (1/\sqrt{3}) + 2 \times (-1) \times (1/\sqrt{3}) + 3 \times 0 = -1/\sqrt{3}$ which gives

$$w_1 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix}$$

and so

$$w_1^{\perp} = \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix}$$

. Its norm is $||w_1^{\perp}|| = \sqrt{8/3}$ and

$$u_2 = \frac{w_1^{\perp}}{||w_1^{\perp}||} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ 0 \end{pmatrix}$$

Then $W_2 = \text{span}\,\{u_1,\ u_2\}$.

Step 3. $v_3 = w_2 + w_2^{\perp}$ where $w_2 =< v_3, u_1 > u_1 + < v_3, u_2 > u_2 \in W_2$ and $w_2^{\perp} = v_3 - w_2$ is in the orthogonal complement of W_2.
Here $< v_3, u_1 >= 0$, $< v_3, u_2 >= 0$ and so $w_2^{\perp} = v_3$. Its norm is $||w_2^{\perp}|| = \sqrt{3}$ and

$$u_3 = \frac{w_2^{\perp}}{||w_2^{\perp}||} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

The orthonormal basis is therefore

$$\{u_1,\ u_2,\ u_3\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}$$

(c) Recall that the standard basis is $\{1,\ x,\ x^2\}$.
Step 1. $||v_1|| = \sqrt{1^2 + 1^2 + 1^2 + 1^2 + 1^2} = \sqrt{5}$. Hence

$$u_1 = \frac{1}{\sqrt{5}}$$

As usual, $W_1 = \text{span}\,\{u_1\}$.

Step 2. $v_2 = w_1 + w_1^{\perp}$ where $w_1 =< v_2, u_1 > u_1 \in W_1$ and $w_1^{\perp} = v_2 - w_1$ is in the orthogonal complement of W_1.
$< v_2, u_1 >= -2 \times \frac{1}{\sqrt{5}} + -1 \times \frac{1}{\sqrt{5}} + 0 \times \frac{1}{\sqrt{5}} + 1 \times \frac{1}{\sqrt{5}} + 2 \times \frac{1}{\sqrt{5}} = 0$ and so $w_1^{\perp} = v_2$.
Its norm is $\sqrt{(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2} = \sqrt{10}$. Hence

$$u_2 = \frac{x}{\sqrt{10}}$$

Again $W_2 = \text{span}\,\{u_1,\ u_2\}$.

Step 3. $v_3 = w_2 + w_2^{\perp}$ where $w_2 =< v_3, u_1 > u_1 + < v_3, u_2 > u_2 \in W_2$ and $w_2^{\perp} = v_3 - w_2$ is in the orthogonal complement of W_2.
$< v_3, u_1 >= (-2)^2 \times \frac{1}{\sqrt{5}} + (-1)^2 \times \frac{1}{\sqrt{5}} + 0^2 \times \frac{1}{\sqrt{5}} + 1^2 \times \frac{1}{\sqrt{5}} + 2^2 \times \frac{1}{\sqrt{5}} = 2\sqrt{5}$ and $< v_3, u_2 >= (-2)^2 \times \frac{(-2)}{\sqrt{10}} + (-1)^2 \times \frac{(-1)}{\sqrt{10}} + 0^2 \times \frac{0}{\sqrt{10}} + 1^2 \times \frac{1}{\sqrt{10}} + 2^2 \times \frac{2}{\sqrt{10}} = 0$. Hence $w_2 = 2$, and so $w_2^{\perp} = x^2 - 2$ which has norm $||w_2^{\perp}|| = \sqrt{(4-2)^2 + (1-2)^2 + (0-2)^2 + (1-2)^2 + (4-2)^2} = \sqrt{14}$. Hence

$$u_3 = \frac{x^2 - 2}{\sqrt{14}}$$

The orthonormal basis is therefore

$$\{u_1,\ u_2,\ u_3\} = \left\{ \frac{1}{\sqrt{5}},\ \frac{x}{\sqrt{10}},\ \frac{x^2-2}{\sqrt{14}} \right\}$$

(d) Step 1. $||v_1||^2 = \int_{-2}^{2} 1^2 dx = 4 \ \Rightarrow ||v_1|| = 2$. Therefore

$$u_1 = \frac{v_1}{||v_1||} = \frac{1}{2}$$

Again $W_1 == \text{span}\,\{u_1\}$.

Step 2. $v_2 = w_1 + w_1^{\perp}$ where $w_1 =< v_2, u_1 > u_1 \in W_1$ and $w_1^{\perp} = v_2 - w_1$ is in the orthogonal complement of W_1.
$< v_2, u_1 >= \int_{-2}^{2} x/2dx = 0$, so $w_1^{\perp} = v_2$. Its norm is $||w_1^{\perp}|| = \left(\int_{-2}^{2} x^2 dx\right)^{1/2} = 4/\sqrt{3}$. Hence

$$u_2 = \frac{x\sqrt{3}}{4}$$

Again $W_2 = \text{span}\,\{u_1,\ u_2\}$.

Step 3. $v_3 = w_2 + w_2^{\perp}$ where $w_2 =< v_3, u_1 > u_1 + < v_3, u_2 > u_2 \in W_2$ and $w_2^{\perp} = v_3 - w_2$ is in the orthogonal complement of W_2.
$< v_3, u_1 >= \int_{-2}^{2} x^2/2dx = 8/3$ while $< v_3, u_2 >= \int_{-2}^{2} x^3\sqrt{3}/dx = 0$. Hence

$$w_2 = 4/3 \qquad \Rightarrow \qquad w_2^{\perp} = x^2 - \frac{4}{3}$$

and $||w_2^{\perp}|| = \left(\int_{-2}^{2} (x^2 - 4/3)^2 dx\right)^{1/2} = 16/3\sqrt{5}$. Therefore

$$u_3 = \frac{\sqrt{5}}{16}\left(3x^2 - 4\right)$$

$W_3 = \text{span}\,\{u_1,\ u_2,\ u_3\}$.

Step 4. $v_4 = w_3 + w_3^{\perp}$ where $w_3 =< v_4, u_1 > u_1 + < v_4, u_2 > u_2 + < v_4, u_3 > u_3 \in W_3$ and $w_3^{\perp} = v_4 - w_3$ is in the orthogonal complement of W_3.
$< v_4, u_1 >= \int_{-2}^{2} x^3/2dx = 0$, $< v_4, u_2 >= \int_{-2}^{2} 4x^4/\sqrt{3}dx = 256/5\sqrt{3}$, and $< v_4, u_3 >= \int_{-2}^{2} \frac{\sqrt{5}}{16}\left(3x^5 - 4x^3\right) dx = 0$. Hence

$$w_3 = \frac{64x}{5} \qquad \Rightarrow \qquad w_3^{\perp} = x^3 - \frac{64x}{5}$$

and $||w_3^{\perp}|| = \left(\int_{-2}^{2} (x^3 - \frac{64x}{5})^2 dx\right)^{1/2} = 8\sqrt{478}/\sqrt{105}$. This gives

$$u_4 = \frac{w_3^{\perp}}{||w_3^{\perp}||} = \sqrt{\frac{105}{478}}\left(\frac{x^3}{8} - \frac{8x}{5}\right)$$

The orthonormal basis is therefore

$$\{u_1,\ u_2,\ u_3,\ u_4\} = \left\{\frac{1}{2},\ \frac{x\sqrt{3}}{4},\ \frac{\sqrt{5}}{16}\left(3x^2-4\right),\ \sqrt{\frac{105}{478}}\left(\frac{x^3}{8}-\frac{8x}{5}\right)\right\}$$

4. We are approximating e^x over $[0,1]$ by a function of the form $a_0+a_1x=a_2x^2$, i.e, an element of P_2. Over the given interval, with the given norm, an orthonormal basis for P_2 is

$$\left\{u_1=1,\ u_2=\sqrt{12}(x-\frac{1}{2}),\ u_3=\sqrt{5}(6x^2-6x+1)\right\}$$

(How was this basis calculated ?)
The least squares approximation is given by

$$p^*(x) = < e^x, u_1 > u_1 + < e^x, u_2 > u_2 + < e^x, u_3 > u_3$$

In this case

$$\begin{aligned} < e^x, u_1 > &= \int_0^1 e^x dx = e-1 \\ < e^x, u_2 > &= \int_0^1 e^x\sqrt{12}(x-\frac{1}{2})dx = \sqrt{12}(\frac{3-e}{2}) \\ < e^x, u_3 > &= \int_0^1 e^x\sqrt{5}(6x^2-6x+1)dx = \sqrt{5}(7e-19) \end{aligned}$$

Hence

$$\begin{aligned} p^*(x) &= (e-1)1 + (\sqrt{12}(\frac{3-e}{2}))(\sqrt{12}(x-\frac{1}{2})) + \sqrt{5}(7e-19)(\sqrt{5}(6x^2-6x+1)) \\ &= 3(13e-35)+12(49-18e)x+30(7e-19)x^2 \\ &\approx 1.013+0.851x+0.839x^2 \end{aligned}$$

5. From Q3 part(c), an orthonormal basis for P_2 is

$$B_2 = \left\{u_1=\frac{1}{\sqrt{5}},\ u_2=\frac{x}{\sqrt{10}},\ u_3=\frac{x^2-2}{\sqrt{14}}\right\}$$

The least squares approximation is given by

$$p_2^*(x) = < f, u_1 > u_1 + < f, u_2 > u_2 + < f, u_3 > u_3$$

In this case

$$\begin{aligned} < f, u_1 > &= \sum_{i=1}^{5} f(x_i)u_1(x_i) = 0\times\frac{1}{\sqrt{5}} + (-1)\times\frac{1}{\sqrt{5}} + 0\times\frac{1}{\sqrt{5}} + 2\times\frac{1}{\sqrt{5}} + 4\times\frac{1}{\sqrt{5}} = \sqrt{5} \\ < f, u_2 > &= 0\times\frac{-2}{\sqrt{10}} + (-1)\times\frac{-1}{\sqrt{10}} + 0\times\frac{0}{\sqrt{10}} + 2\times\frac{1}{\sqrt{10}} + 4\times\frac{2}{\sqrt{10}} = \frac{11}{\sqrt{10}} \\ < f, u_3 > &= 0\times\frac{2}{\sqrt{14}} + (-1)\times\frac{-1}{\sqrt{14}} + 0\times\frac{-2}{\sqrt{14}} + 2\times\frac{-1}{\sqrt{14}} + 4\times\frac{2}{\sqrt{14}} = \frac{7}{\sqrt{14}} \end{aligned}$$

Hence

$$\begin{aligned} p_2^*(x) &= \sqrt{5}\frac{1}{\sqrt{5}} + \frac{11}{\sqrt{10}}\frac{x}{\sqrt{10}} + \frac{7}{\sqrt{14}}\frac{x^2-2}{\sqrt{14}} \\ &= \frac{11}{10}x + \frac{1}{2}x^2 \end{aligned}$$

From the answer to Q3 part (c), we can infer that $B_1 = \{u_1,\ u_2\}$ is a basis for P_1, where u_1 and u_2 are as defined above - (why ?). Hence proceeding in analogous fashion , we get

$$\begin{aligned} p_1^*(x) &= < f, u_1 > u_1 + < f, u_2 > u_2 \\ &= \sqrt{5}\frac{1}{\sqrt{5}} + \frac{11}{\sqrt{10}}\frac{x}{\sqrt{10}} \\ &= 1 + \frac{11}{10}x \end{aligned}$$