Let $Y(s) = \mathcal{L}[y(t)]$.

1. (a) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [sY(s) - y(0)] + Y(s) &= \frac{1}{s^2} - \frac{3}{s} \\ \Rightarrow (s+1)Y(s) - 2 &= \frac{1}{s^2} - \frac{3}{s} \\ \Rightarrow Y(s) &= \frac{1 - 3s + 2s^2}{s^2(s+1)} \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = \frac{6}{s+1} + \frac{1}{s^2} - \frac{4}{s}$$

Taking the inverse transform yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \left(6e^{-t} + t - 4\right) u_0(t) \end{aligned}$$

(b) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [sY(s) - y(0)] - 4Y(s) &= 3 \times \frac{2}{s^2 + 2^2} \\ \Rightarrow (s-4)Y(s) &= \frac{6}{s^2+4} \\ \Rightarrow Y(s) &= \frac{6}{(s-4)(s^2+4)} \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = \frac{3/10}{s-4} - \frac{(3/10)\,s}{s^2+4} - \frac{(6/10) \times 2}{s^2+4}$$

Taking the inverse transform yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \left(\frac{3}{10}e^{4t} - \frac{3}{10}\cos 2t - \frac{6}{10}\sin 2t\right) u_0(t) \end{aligned}$$

(c) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [sY(s) - y(0)] + 3Y(s) &= 6 \times \frac{1}{s}e^{-s} \\ \Rightarrow (s+3)Y(s) - 1 &= \frac{6}{s}e^{-s} \\ \Rightarrow Y(s) &= \frac{1}{s+3} + \frac{6}{s(s+3)}e^{-s} \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = \frac{1}{s+3} + \left(\frac{2}{s} - \frac{2}{s+3}\right)e^{-s}$$

Taking the inverse transform yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= e^{-3t}u_0(t) + \left(2 - 2e^{-3(t-1)}\right) u_1(t) \end{aligned}$$

(d) Noting that $f(t) = u_0(t) - 2u_1(t) + u_2(t)$, we can proceed as follows: Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [sY(s) - y(0)] + 2Y(s) &= \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) \\ \Rightarrow (s+2)Y(s) - 1 &= \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) \\ \Rightarrow Y(s) &= \frac{1}{s+2} + \frac{1}{s(s+2)}\left(1 - 2e^{-s} + e^{-2s}\right) \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = \frac{1}{s+2} + \frac{1}{2}(\frac{1}{s} - \frac{1}{s+2})\left(1 - 2e^{-s} + e^{-2s}\right)$$

Taking the inverse transform yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= e^{-2t}u_0(t) + \frac{1}{2}(1 - e^{-2t})u_0(t) - (1 - e^{-2(t-1)})u_1(t) + \frac{1}{2}(1 - e^{-2(t-2)})u_2(t) \\ &= \frac{1}{2}(1 + e^{-2t})u_0(t) - (1 - e^{-2(t-1)})u_1(t) + \frac{1}{2}(1 - e^{-2(t-2)})u_2(t) \end{aligned}$$

(e) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [s(sY(s) - y(0)) - \frac{dy}{dt}(0)] + 4[sY(s) - y(0)] + 3Y(s) &= \frac{1}{s^2} + \frac{2}{s} \\ \Rightarrow (s^2 + 4s + 3)Y(s) - 1 &= \frac{1}{s^2} + \frac{2}{s} \\ \Rightarrow Y(s) &= \frac{1}{(s+1)(s+3)} + \frac{1}{s^2(s+1)(s+3)} \\ &\quad + \frac{2}{s(s+1)(s+3))} \\ &= \frac{(s+1)^2}{s^2(s+1)(s+3)} \\ &= \frac{s+1}{s^2(s+3)} \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = -\frac{2/9}{s+3} + \frac{1/3}{s^2} + \frac{2/9}{s}$$

Taking the inverse transform yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \left(-\frac{2}{9}e^{-3t} + \frac{1}{3}t + \frac{2}{9}\right)u_0(t) \end{aligned}$$

(f) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [s(sY(s) - y(0)) - \frac{dy}{dt}(0)] + 4Y(s) &= \frac{1}{s-2} \\ \Rightarrow (s^2 + 4)Y(s) - 2s &= \frac{1}{s-2} \\ \Rightarrow Y(s) &= \frac{2s}{s^2+4} + \frac{1}{(s-2)(s^2+4)} \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = \frac{1/8}{s-2} - \frac{1/4}{s^2+4} + \frac{(15/8)s}{s^2+4}$$

Taking the inverse transform yields

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \left(\frac{1}{8}e^{2t} - \frac{1}{8}\sin 2t + \frac{15}{8}\cos 2t\right)u_0(t) \\ &= \left(\frac{1}{8}e^{2t} + \frac{\sqrt{226}}{8}\cos(2t+\phi)\right)u_0(t) \end{aligned}$$

where $\phi = \arctan\left(\frac{1}{15}\right)$.

(g) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [s(sY(s) - y(0)) - \frac{dy}{dt}(0)] + 2[sY(s) - y(0)] + Y(s) &= \frac{1}{s+1} \\ \Rightarrow (s^2 + 2s + 1)Y(s) &= \frac{1}{s+1} \\ \Rightarrow Y(s) &= \frac{1}{(s+1)^3} \end{aligned}$$

Using the 2nd Shift theorem $F(s+a) = \mathcal{L}[e^{at}f(t)]$ gives on taking the inverse transform

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \frac{1}{2}e^{-t}t^2 u_0(t) \end{aligned}$$

(h) Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [s(sY(s) - y(0)) - \frac{dy}{dt}(0)] + 4[sY(s) - y(0)] + 4Y(s) &= 4 \times \frac{1}{s} \\ \Rightarrow (s^2 + 4s + 4)Y(s) - 2s - 8 &= \frac{4}{s} \\ \Rightarrow Y(s) &= \frac{2s^2 + 8s + 4}{s(s+2)^2} \end{aligned}$$

expanding the right hand side using partial fractions

$$\Rightarrow Y(s) = \frac{1}{s} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

Using the 2nd Shift theorem gives on taking the inverse transform

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] \\ &= \left(1 + e^{-2t} + 2te^{-2t}\right)u_0(t) \end{aligned}$$

2. (a)

$$\begin{aligned} 1 * \sin 2t &= \int_0^t 1 \sin 2(t-u)du \\ &= \frac{1}{2}\cos 2(t-u)\Big|_0^t \\ &= \frac{1}{2}(1 - \cos 2t) \end{aligned}$$

Alternatively

$$\begin{aligned} 1 * \sin 2t &= \mathcal{L}^{-1}[\mathcal{L}[1]][\mathcal{L}[\sin 2t]] \\ &= \mathcal{L}^{-1}[\frac{1}{s} \times \frac{2}{s^2 + 2^2}] \\ &= \mathcal{L}^{-1}[\frac{1/2}{s} - \frac{1/2s}{s^2 + 2^2}] \\ &= \frac{1}{2} - \frac{\cos 2t}{2} \end{aligned}$$

(b)

$$\begin{aligned} e^{-t} * 1 &= \int_0^t e^{-u} du \\ &= -e^{-u} \Big|_0^t \\ &= 1 - e^{-t} \end{aligned}$$

Alternatively

$$\begin{aligned} e^{-t} * 1 &= \mathcal{L}^{-1}[\mathcal{L}[e^{-t}]][\mathcal{L}[1]] \\ &= \mathcal{L}^{-1}[\frac{1}{s+1} \times \frac{1}{s}] \\ &= \mathcal{L}^{-1}[\frac{1}{s} - \frac{1}{s+1}] \\ &= 1 - e^{-t} \end{aligned}$$

(c)

$$\begin{aligned} t * t &= \int_0^t u(t-u) du \\ &= \frac{u^2 t}{2} - \frac{u^3}{3} \Big|_0^t \\ &= \frac{t^3}{6} \end{aligned}$$

Alternatively

$$\begin{aligned} t * t &= \mathcal{L}^{-1}[\mathcal{L}[t]][\mathcal{L}[t]] \\ &= \mathcal{L}^{-1}[\frac{1}{s^2} \times \frac{1}{s^2}] \\ &= \mathcal{L}^{-1}[\frac{1}{s^4}] = \mathcal{L}^{-1}[\frac{3!}{6s^4}] \\ &= \frac{t^3}{6} \end{aligned}$$

(d)

$$\begin{aligned} t * e^{2t} &= \int_0^t u e^{2(t-u)} du \\ &= -\frac{2u+1}{4} e^{2(t-u)} \Big|_0^t \\ &= \frac{1}{4} e^{2t} - \frac{2t+1}{4} \end{aligned}$$

Alternatively

$$\begin{aligned} t * e^{2t} &= \mathcal{L}^{-1}[\mathcal{L}[t]][\mathcal{L}[e^{2t}]] \\ &= \mathcal{L}^{-1}[\frac{1}{s^2} \times \frac{1}{s-2}] \\ &= \mathcal{L}^{-1}[-\frac{1/4}{s} - \frac{1/2}{s^2} + \frac{1/4}{s-2}] \\ &= -\frac{1}{4} - \frac{t}{2} + \frac{1}{4} e^{2t} \end{aligned}$$

3. Taking the *Laplace* transform of the differential equation gives

$$\begin{aligned} [sY(s) - y(0)] + 3Y(s) &= \mathcal{L}[\cos 4t] \\ \Rightarrow (s+3)Y(s) &= \mathcal{L}[\cos 4t] \\ \Rightarrow Y(s) &= \frac{\mathcal{L}[\cos 4t]}{s+3} \\ \Rightarrow Y(s) &= \mathcal{L}[\cos 4t]\mathcal{L}[e^{-3t}] \\ \Rightarrow y(t) &= \cos 4t * e^{-3t} \end{aligned}$$

Now

$$\begin{aligned}
y(t) &= \int_0^t \cos 4u e^{-3(t-u)} du \\
&= \Re\left(\int_0^t e^{\imath 4u} e^{-3(t-u)} du\right) \\
&= \Re\left(e^{-3t} \int_0^t e^{(3+\imath 4)u} du\right) \\
&= \Re\left(e^{-3t} \frac{e^{(3+\imath 4)u}}{3+\imath 4} \Big|_0^t\right) \\
&= \Re\left(e^{-3t} \frac{3-\imath 4}{25} \left(e^{3t}(\cos 4t + \imath \sin 4t) - 1\right)\right) \\
&= \frac{3\cos 4t + 4\sin 4t - 3e^{-3t}}{25}
\end{aligned}$$

4. Taking the *Laplace* transform of the integral equation gives

$$\begin{aligned}
Y(s) &= \frac{1}{s^2+1^2} + 2\frac{s}{s^2+1^2} Y(s) \\
\Rightarrow (1 - \frac{2s}{s^2+1})Y(s) &= \frac{1}{s^2+1} \\
\Rightarrow \frac{(s-1)^2}{s^2+1} Y(s) &= \frac{1}{s^2+1} \\
\Rightarrow Y(s) &= \frac{1}{(s-1)^2}
\end{aligned}$$

Using the 2nd Shift theorem gives

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}[Y(s)] \\
&= te^t u_0(t)
\end{aligned}$$