UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4003

SEMESTER: Autumn 2015/16

MODULE TITLE: Engineering Mathematics 3

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**
A table of *Laplace* transforms is attached.

Table of Laplace Transforms

$f(t),\ t \geq 0$	$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{1}{a-b}(e^{at} - e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$\frac{a}{a-b}e^{at} - \frac{b}{a-b}e^{bt}$	$\frac{s}{(s-a)(s-b)}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}F(s)$
$e^{at}f(t)$	$F(s-a)$
Heaviside $u_a(t)$	$\frac{e^{-as}}{s}$
$f(t-a)u_a(t)$	$e^{-as}F(s)$
Ramp $R(t-a)$	$\frac{e^{-as}}{s^2}$
$tf(t)$	$-F'(s)$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\,d\sigma$
$(f * g)(t) \equiv \displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
$f(t) = f(t+p)$	$\dfrac{1}{1-e^{-sp}} \displaystyle\int_0^p f(t)e^{-st}\,dt$

 Marks

1. (a) Sketch the graphs of the following functions $f(t), t \geq 0$ and find their *Laplace* transforms:

   (i) $f(t) = \begin{cases} 0, & \text{if } t < 3 \\ e^{-2t+6}, & \text{if } 3 \leq t \end{cases}$ 2

   (ii) $f(t) = t\cos 2t$ 3

   (iii) $f(t) = \begin{cases} 1, & \text{if } t < 1 \\ 2-t, & \text{if } 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$ 3

   (b) Find the inverse *Laplace* transforms of the following 3,2,3

   $(i)\ \dfrac{2s+3}{s^2+5s+6} \qquad (ii)\ \dfrac{2s+3}{s^2+5s+6}e^{-s} \qquad (iii)\ \ln(s^3+s)$

2. (a) Use *Laplace* transforms to solve the initial value problem 9

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 6t + 5, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = -3$$

   (b) Use *Laplace* transforms to solve the initial value problem 7

$$\frac{dy}{dt} = 3y - 4\int_0^t e^{-(t-u)}y(u)\,du, \qquad y(0) = 1$$

3. (a) Sketch the graph of the following periodic function and find its *Fourier* series 8

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leq -\frac{\pi}{2} \\ 0, & \text{if } -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ 1, & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

   (b) Use the *Fourier* series of part (a) to find a particular solution of the differential equation 6

$$\frac{d^2y}{dx^2} + 2y = f(x)$$

   where $f(x)$ is as in part (a).

   (c) The function $g : \mathbb{R} \to \mathbb{R}$ is an odd function of period 2π with *Fourier* series $\sum_{n=1}^{\infty} b_n \sin(nx)$. The function $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = 4 + g\left(\frac{x}{3}\right)$$

   Find the *Fourier* series of h. 2

4. (a) Determine whether the set of vectors

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \right\}$$

forms a basis for $\mathbb{R}^3$. 6

(b) Define span S. What is the dimension of span S ? 4

(c) Is $(3,\ 2,\ 1)^T \ \in$ span S ?
For what value of a is $(2,\ a,\ -2)^T \in$ span S ? 6

5. (a) Let P_1 be the space of polynomials of degree at most one. Use the *Gram-Schmidt* process to transform the standard basis $\{1,\ x\}$ for P_1 to the orthonormal one defined by the inner product

$$< p, q >= \sum_{i=1}^{5} p(x_i)q(x_i)$$

where $x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = 3,\ x_5 = 4.$ 8

(b) Let f be a function on $[0, 4]$, taking the value $f(x_i)$ at $x = x_i,\ i = 1, 2, 3, 4, 5$ as given in the following table

x_i	$f(x_i)$
0	-2
1	-1
2	2
3	3
4	5

Find the least squares approximation to f in P_1 using the inner product defined in part (a). 8

6. Find the eigenvalues of the matrix 4

$$A = \begin{pmatrix} -1 & 4 & -4 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

Show that

$$P = \begin{pmatrix} -2 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

diagonalises A. 6
Hence or otherwise solve the system of differential equations 6

$$\frac{d}{dt}X = AX, \qquad X(0) = (2, 0, -1)^T$$

7. (a) Find the *Cholesky* decomposition ($A = LL^T$, where L is lower triangular) of the matrix 5

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & -4 \\ -2 & -4 & 26 \end{pmatrix}$$

and hence solve the system of equations $Ax = b$ where 3

$$b = \begin{pmatrix} 8 \\ 10 \\ -54 \end{pmatrix}$$

(b) The system of equations $Ax = b$ can also be solved using an iterative scheme. One such scheme for a square matrix A is the *Jacobi* method in which successive approximations are generated by

$$x_{k+1} = D^{-1}b - D^{-1}\tilde{A}x_k$$

where x_k is the k-th approximation, and $A = D + \tilde{A}$ with D being a diagonal matrix whose entries are the respective diagonal entries of A, and $\tilde{A} = A - D$. Show how the iteration scheme may be derived. Under what condition does the scheme converge for any initial approximation x_0 ? 4

Using the fact that the roots of the equation

$$\lambda^3 - (11/13)\lambda + (2/13) = 0$$

do not satisfy $|\lambda| < 1$, does the scheme converge for the A matrix of part (a)? Explain your answer. 4