UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4003

SEMESTER: Autumn 2014/15

MODULE TITLE: Engineering Mathematics 3

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions.**
A table of *Laplace* transforms is attached.

Table of Laplace Transforms

$f(t),\ t \geq 0$	$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{1}{a-b}(e^{at} - e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$\frac{a}{a-b}e^{at} - \frac{b}{a-b}e^{bt}$	$\frac{s}{(s-a)(s-b)}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s}F(s)$
$e^{at}f(t)$	$F(s-a)$
Heaviside $u_a(t)$	$\frac{e^{-as}}{s}$
$f(t-a)u_a(t)$	$e^{-as}F(s)$
Ramp $R(t-a)$	$\frac{e^{-as}}{s^2}$
$tf(t)$	$-F'(s)$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\, d\sigma$
$(f * g)(t) \equiv \displaystyle\int_0^t f(t-\tau)g(\tau)\, d\tau$	$F(s)G(s)$
$f(t) = f(t+p)$	$\dfrac{1}{1-e^{-sp}} \displaystyle\int_0^p f(t)e^{-st}\, dt$

1. (a) Sketch the graphs of the following functions $f(t), t \geq 0$ and find their *Laplace* transforms:

   (i) $f(t) = te^{-2t}$ 2

   (ii) $f(t) = \begin{cases} 0, & \text{if } t < 1 \\ \sin \pi(t-1), & \text{if } 1 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}$ 3

   (iii) $f(t) = \begin{cases} 3t, & \text{if } 0 \leq t < 1 \\ 6 - 3t, & \text{if } 1 \leq t < 2 \end{cases}$ $\qquad f(t+2) = f(t)$ 3

   (b) Find the inverse *Laplace* transforms of the following 2,3,3

   $$(i) \quad \frac{2s+2}{s^2 - 7s + 6} \qquad (ii) \quad \frac{e^{-s}}{s^2 - 3s + 2} \qquad (iii) \quad \arctan\left(\frac{s+3}{2}\right)$$

2. (a) Use *Laplace* transforms to solve the initial value problem 8

   $$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 4e^{-t}, \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 3$$

   (b) Prove that

   $$\mathcal{L}\left(\int_0^t y(u)\, du\right) = \frac{1}{s}\mathcal{L}\left(y(t)\right)$$

   2

   (c) Use *Laplace* transforms to solve the initial value problem 6

   $$\frac{dy}{dt} + 2y + \int_0^t y(u)\, du = 0, \qquad y(0) = 2$$

3. (a) Sketch the graph of the following periodic function and find its *Fourier* series 8

   $$f(x) = \begin{cases} -x - \frac{\pi}{2}, & \text{if } -\pi < x \leq -\frac{\pi}{2} \\ 0, & \text{if } -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ x - \frac{\pi}{2}, & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

   In particular show that

   $$a_n = \frac{2}{\pi n^2}\left(\cos n\pi - \cos(n\pi/2)\right), \qquad n > 0$$

   (b) Hence evaluate the series 4

   $$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$$

   (c) The function $g : \mathbb{R} \to \mathbb{R}$ is an even function of period 2π with *Fourier* series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$. The function $h : \mathbb{R} \to \mathbb{R}$ is defined by

   $$h(x) = -\frac{a_0}{2} + g\left(\frac{x}{2}\right)$$

   Find the *Fourier* series of h. 4

4. (a) Define the row rank, column rank and rank of $A \in \mathbb{R}^{m \times n}$. 2

(b) For

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & -5 & 6 \\ 3 & -3 & -2 \\ 4 & -5 & -2 \end{pmatrix}$$

find its rank ($= r$). 6

(c) Choose columns from the matrix A of part (b) that form a basis for its column space. Write any remaining columns as a linear combination of the basis vectors. 2

Let $C \in \mathbb{R}^{m \times r}$ be the matrix whose columns are the basis vectors. Hence show that there exists $R \in \mathbb{R}^{r \times n}$ such that

$$A = CR.$$

. 2

What does this say about the relationship between the rows of A and rows of R, and what does this in turn say about the row rank and column rank? 2

Can the converse be established between the column rank and row rank? And if so, what conclusion follows? 2

5. (a) Let P_2 be the space of polynomials of degree at most two. Use the *Gram-Schmidt* process to transform the standard basis $\{1,\ x,\ x^2\}$ for P_2 to the orthonormal one defined by the inner product

$$< p, q >= \int_{-1}^{1} p(x)q(x)\, dx$$

8

(b) Find the least squares approximation to e^x over the interval $[-1,\ 1]$ in P_2 using the inner product defined in part (a). 8

6. Find the eigenvalues and corresponding eigenspaces of the matrix 6

$$A = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$$

Show that A is diagonalisable over the reals and find the matrix P which diagonalises it. 4

Hence or otherwise solve the system of differential equations 6

$$\frac{d}{dt}X = AX, \qquad X(0) = (6, 1)^T$$

7. (a) Find the *Cholesky* decomposition ($A = \tilde{L}\tilde{L}^T$, where $\tilde{L}$ is lower triangular) of the matrix 5

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 10 & 4 \\ 1 & 4 & 6 \end{pmatrix}$$

and hence solve the system of equations $Ax = b$ where 3

$$b = \begin{pmatrix} 3 \\ 8 \\ -6 \end{pmatrix}$$

(b) The system of equations $Ax = b$ can also be solved using an iterative scheme. One such scheme for a square matrix A is the *Gauss - Seidel* method in which successive approximations are generated by

$$x_{k+1} = (L + D)^{-1}b - (L + D)^{-1}Ux_k$$

where x_k is the k-th approximation, and $L + D + U = A$ with D being a diagonal matrix whose entries are the respective diagonal entries of A, and L and U being lower triangular and upper triangular matrices containing the entries of A below and above the diagonal respectively. Show how the iteration scheme for generating the successive approximations may be derived. Under what condition does this scheme converge for any initial approximation x_0 ? 4

Does the scheme converge for the A matrix of part (a)? Explain your answer. 4