1 Solving Linear Recurrences

It is straightforward to verify that the recurrence

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \mathbf{u}_k, \qquad \mathbf{x}_0 \text{ given} \tag{1}$$

where $\mathbf{x}_k$ is a n-vector, A a $n \times n$ constant matrix and $\mathbf{u}_k$ an externally applied input, has solution

$$\mathbf{x}_k = A^k \mathbf{x}_0 + \sum_{j=0}^{k-1} A^{k-1-j} \mathbf{u}_j \tag{2}$$

For autonomous Dynamical Systems, the external input is zero and thus the solution of

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \qquad \mathbf{x}_0 \text{ given} \tag{3}$$

is

$$\mathbf{x}_k = A^k \mathbf{x}_0 = P^{-1} J^k P \mathbf{x}_0 = \sum_{i=1}^{m} \lambda_i^k \mathbf{g}_i(k) \tag{4}$$

where J is the *Jordan* form of A (see Review of Linear Algebra Topics); m is the number of distinct eigenvalues (λ_i) and $\mathbf{g}_i$ is a polynomial vector in k of degree at most $n-1$. If A is diagonalisable, then $\mathbf{g}_i$ is a constant vector.