Linearisation

It is frequently advantageous to approximate nonlinear models by linear ones. Consider the continuous-time state model with given initial state

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t\,; \mathbf{a}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

When the parameter has nominal value $\mathbf{a}^*$, the solution of Eq(1) denoted by $\mathbf{x}^*(t)$ defined on the interval , $t_0 \le t < t_f$ satisfies

$$\dot{\mathbf{x}}^*(t) = \mathbf{f}(\mathbf{x}^*(t), t\,; \mathbf{a}^*) \qquad \mathbf{x}^*(t_0) = \mathbf{x}_0 \tag{2}$$

Let us perturb the nominal value of the parameter as follows:

$$\mathbf{a} = \mathbf{a}^* + \delta\mathbf{a}$$

where $\delta\mathbf{a}$ is "small". The corresponding state trajectory is perturbed to

$$\mathbf{x} = \mathbf{x}^* + \delta\mathbf{x}$$

Since these perturbed variables satisfy the state equation, we get

$$\dot{\mathbf{x}}^* + \dot{\delta\mathbf{x}} = \mathbf{f}(\mathbf{x}^* + \delta\mathbf{x}, t\,; \mathbf{a}^* + \delta\mathbf{a}) \qquad \mathbf{x}^*(t_0) + \delta\mathbf{x}(t_0) = \mathbf{x}_0 + \delta\mathbf{x}_0$$

Expanding $\mathbf{f}$ in a *Taylor* series about $(\mathbf{x}^*,\ \mathbf{a}^*)$ yields

$$\dot{\mathbf{x}}^* + \dot{\delta\mathbf{x}} = \mathbf{f}(\mathbf{x}^*, t\,; \mathbf{a}^*) + A\,\delta\mathbf{x} + B\,\delta\mathbf{a} + \mathrm{O}(\delta\mathbf{x}, \delta\mathbf{a}) \qquad \mathbf{x}^*(t_0) + \delta\mathbf{x}(t_0) = \mathbf{x}_0 + \delta\mathbf{x}_0$$

where the "Jacobian" matrices ($M = [m_{ij}]$)are defined by

$$A(t) = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{x}^*, \mathbf{a}^*)\right] \qquad B(t) = \left[\frac{\partial f_i}{\partial a_j}(\mathbf{x}^*, \mathbf{a}^*)\right]$$

and $\mathrm{O}(\delta\mathbf{x}, \delta\mathbf{a})$ represents higher order terms in $\delta\mathbf{x}$ and $\delta\mathbf{a}$.
Using Eq(2) this simplifies to

$$\dot{\delta\mathbf{x}} = A\,\delta\mathbf{x} + B\,\delta\mathbf{a} + \mathrm{O}(\delta\mathbf{x}, \delta\mathbf{a}) \qquad \delta\mathbf{x}(t_0) = \delta\mathbf{x}_0$$

Furthermore, if $(\delta\mathbf{x}, \delta\mathbf{a})$ are "small enough" to neglect the higher order terms, then we get (at least in some neighbourhood of $(\mathbf{x}^*, \mathbf{a}^*)$) the linearised model

$$\dot{\delta\mathbf{x}} = A\,\delta\mathbf{x} + B\,\delta\mathbf{a} \qquad \delta\mathbf{x}(t_0) = \delta\mathbf{x}_0 \tag{3}$$

Solving Eq(3) for $\delta\mathbf{x}(t)$, we obtain(at least in some neighbourhood of $(\mathbf{x}^*, \mathbf{a}^*)$) that

$$\mathbf{x}(t) \approx \mathbf{x}^*(t) + \delta\mathbf{x}(t)$$

One of the more common applications of Linearisation is in dealing with the stability of equilibrium states. If a system is in equilibrium at $\mathbf{x}_E$ corresponding to nominal parameter $\mathbf{a}^*$, then its state does not change; so the state equation become

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_E, t\,; \mathbf{a}^*) \tag{4}$$

Given $\mathbf{a}^*$, we can solve Eq(4) to get $\mathbf{x}_E$. The linearised model (Eq(3)) is then calculated with $\mathbf{x}^* \equiv \mathbf{x}_E$.

Example: The nonlinear inverted pendulum model described in Elgerd: *Control Systems Theory* (McGraw Hill 1967), Chapter 2, is, after elimination of some variables

$$\begin{aligned}\left(I + mL^2\sin^2\phi + \frac{m}{m+M}ML^2\cos^2\phi\right)\ddot{\phi} + \left(\frac{m}{m+M}mL^2\sin\phi\cos\phi\right)\dot{\phi}^2 \\ = \quad mgL\sin\phi - \left(\frac{m}{m+M}L\cos\phi\right)a\end{aligned}$$

An equivalent state model is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b(x_1)}{e(x_1)}x_2^2 + \frac{c(x_1)}{e(x_1)} - \frac{d(x_1)}{e(x_1)}a\end{aligned}$$

where $x_1 = \phi,\ x_2 = \dot{\phi}$ and

$$\begin{aligned}e(x_1) &= I + mL^2\sin^2 x_1 + \frac{m}{m+M}ML^2\cos^2 x_1 \\ b(x_1) &= \frac{m}{m+M}mL^2\sin x_1\cos x_1 \\ c(x_1) &= mgL\sin x_1 \\ d(x_1) &= \frac{m}{m+M}L\cos x_1\end{aligned}$$

There is an equilibrium state corresponding to $a \equiv 0$ at $\phi(t) \equiv 0$. Thus we set

$$a^* \equiv 0 \qquad \mathbf{x}^*(t) \equiv 0$$

Then the linearised model (with $\delta\mathbf{x} = \mathbf{x},\ \delta\mathbf{a} = a$) is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g(m+M)mL}{I(m+M)+mML^2}x_1 - \frac{mL}{I(m+M)+mML^2}a\end{aligned}$$

or in vector-matrix notation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -\beta \end{bmatrix}a$$

with $\alpha = [g(m+M)mL]\,/\,[I(m+M)+mML^2]$ and $\beta = mL/\,[I(m+M)+mML^2]$. □

The Linearisation process is analogous for discrete-time models.