Homework : Mathematical Induction

1. For all $n \in \mathbb{N}$, prove that

   (a) 5 divides $n^5 - n$.

   (b)

$$1 \times 2 \quad + \quad 2 \times 3 \quad + \quad 3 \times 4 \quad + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   (c)

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

2. Guess a formula for the given sum and then use mathematical induction to prove it.

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)}$$

   Hint: What you could do is evaluate the expression for (say) $n = 1, 2, 3, ...$ and from these few infer what the formula should be. This procedure is akin to what experimental scientists do ("scientific induction"); it is not mathematical induction, which is a proof method.

3. Use mathematical induction to show that n straight lines divide the plane into $(n^2 + n + 2)/2$ regions. Assume that no two lines are parallel, and that no three lines have a common point.

4. Solve the following recurrence relations by iterating (scientific induction), then prove your answers using mathematical induction:

   (a)

$$a_{n+1} = 2a_n, \qquad a_1 = 5$$

   (b)

$$b_{n+1} = -\frac{1}{4}b_n, \qquad b_1 = 4$$

   What type of sequences are these?

5. In Q 4, both solutions of the recurrences turned out to be of the form Ar^n for some "geometric ratio" r and constant A. Motivated by this let's make the ansatz ("educated guess") that the solution of

$$c_{n+2} = 5c_{n+1} - 4c_n$$

   can be given by $c_n = Ar^n$. [Note that $r \equiv 0$ works].

   (a) Show that there are two non-zero values for r (which I'll call r_1 and r_2 below) which work, and find them.

   (b) Show that $c_n = A_1 r_1^n$ and $c_n = A_2 r_2^n$, where A_1 and A_2 are arbitrary constants, both satisfy the recurrence relation;

(c) if in addition to the recurrence relation we require that $c_1 = 6$ and $c_2 = 18$, then neither of the solutions of part 5b satisfy both of these initial values.

(d) but a combination of the two solutions $c_n = A_1 r_1^n + A_2 r_2^n$ can be made to satisfy the initial values by an appropriate choice for A_1 and A_2.