3.4 Concepts of Solutions to 2-Player Matrix Games - Minimax Solutions

The *minimax strategy* of a player is the strategy that maximises his/her expected payoff under the assumption that the other player aims to minimise this payoff.

The *minimax payoff* of a player is the maximum expected payoff that he/she can guarantee him/hirself.

In zero-sum games such a concept seems very reasonable, since by minimising the expected reward of the opponent, an individual maximises his/her expected reward.

Minimax Solutions

One way of finding a player's minimax strategy is to use linear programming.

Suppose players choose one of two actions. In this case you can think of the strategy of a player as the probability of playing the first action, i.e. choice from the interval $[0, 1]$. The simplest method of solution is to use calculus.

Denote the reward of Player i when Player 1 uses strategy p and Player 2 uses strategy q by $R_i(p, q)$.

Minimax Solutions - Example 3.4.1

Find the minimax strategies and payoffs in the following matrix game

	A	B
A	(5,3)	(2,6)
B	(3,2)	(4,5)

Minimax Solutions - Example 3.4.1

Suppose Player 1 takes action A with probability p and Player 2 takes action A with probability q. The expected reward of Player 1 is given by

$$R_1(p,q) = 5pq + 2p(1-q) + 3(1-p)q + 4(1-p)(1-q) = 4pq - 2p - q + 4.$$

Player 1 assumes that Player 2 will minimise her payoff. In order to derive which strategy of Player 2 minimises Player 1's payoff, we calculate $\frac{\partial R_1(p,q)}{\partial q}$ (i.e. the rate of change of Player 1's expected payoff with respect to the strategy of Player 2).

$$\frac{\partial R_1(p,q)}{\partial q} = 4p - 1.$$

In general, this derivative will be a linear function of p.

Minimax Solutions - Example 3.4.1

There are three possibilities

1. $\frac{\partial R_1(p,q)}{\partial q} > 0$ for all $p \in [0,1]$. In this case to minimise Player 1's payoff, Player 2 should play $q = 0$ (i.e. always take his second action). The minimax strategy of Player 1 is to take the action that maximises her reward when Player 2 takes his second action.
2. $\frac{\partial R_1(p,q)}{\partial q} < 0$ for all $p \in [0,1]$. In this case to minimise Player 1's payoff, Player 2 should play $q = 1$ (i.e. always take his first action). The minimax strategy of Player 1 is to take the action that maximises her reward when Player 2 takes his first action.
3. There exists some $p \in [0,1]$ such that $\frac{\partial R_1(p,q)}{\partial q} = 0$. When Player 1 uses this strategy, her expected payoff does not depend on the strategy used by Player 2. This is her minimax strategy.

Minimax Solutions - Example 3.4.1

$$\frac{\partial R_1(p,q)}{\partial q} = 0 \Rightarrow p = \frac{1}{4}.$$

We have

$$R_1(\frac{1}{4}, q) = 4 - 2p = 3.5.$$

It follows that the minimax strategy of Player 1 is to choose action A with probability 0.25. In this way she guarantees herself a payoff of 3.5

Minimax Solutions - Example 3.4.1

It should be noted that if Player 1 chose $p > \frac{1}{4}$, then $\frac{\partial R_1(p,q)}{\partial q} > 0$ and Player 2 minimises Player 1's reward by choosing $q = 0$, i.e. Playing B.

In this case the payoff of Player 1 would be $R_1(p, 0) = 4 - 2p < 3.5$.

Similarly, if Player 1 chose $p < \frac{1}{4}$, then $\frac{\partial R_1(p,q)}{\partial q} < 0$ and Player 2 minimises Player 1's reward by choosing $q = 1$, i.e. Playing A.

In this case the payoff of Player 1 would be $R_1(p, 1) = 2p + 3 < 3.5$. It is thus clear that 3.5 is the maximum expected payoff that Player 1 can guarantee herself.

Minimax Solutions - Example 3.4.1

We can calculate the minimax strategy of Player 2 in an analogous way, by deriving his expected payoff as a function of p and q and differentiating with respect to p, the strategy of Player 1. We have

$$R_2(p,q) = 3pq + 6p(1-q) + 2(1-p)q + 5(1-p)(1-q) = p - 3q + 5.$$

Hence,

$$\frac{\partial R_2(p,q)}{\partial p} = 1 > 0.$$

It follows that Player 1 always minimises Player 2's expected reward by playing $p = 0$, i.e. always taking action B.

Minimax Solutions - Example 3.4.1

If Player 1 takes action B, then Player 2 should take action B. This ensures him a payoff of 5.

Note: It can be seen from the payoff matrix that if Player 1's aim is to minimise Player 2's payoff, then her action B dominates action A.

This is due to the fact that Player 2's reward is always smaller when Player 1 plays B, whatever action Player 2 takes.

Minimax Solutions - Example 3.4.1

Suppose both players use their minimax strategies, i.e. Player 1 chooses action A with probability 0.25 and Player 2 always chooses action B.

The payoff of Player 1 is
$R_1(0.25A + 0.75B, B) = 0.25 \times 2 + 0.75 \times 4 = 3.5$.

The payoff of Player 2 is
$R_2(0.25A + 0.75B, B) = 0.25 \times 6 + 0.75 \times 5 = 5.25$, i.e. the reward obtained by Player 2 is greater than his minimax payoff.

In general, if a player's minimax strategy is a pure strategy, when both players play their minimax strategies, he/she may obtain a greater expected payoff than his/her minimax payoff.

Concepts of Solutions to 2-Player Matrix Games - Pure Nash equilibria

A pair of actions (A^*, B^*) is a pure Nash equilibrium if

$$R_1(A^*, B^*) \geq R_1(A, B^*); \qquad R_2(A^*, B^*) \geq R_2(A^*, B).$$

for any action A available to Player 1 and any action B available to Player 2.

That is to say that a pair of actions is a Nash equilibrium if neither player can gain by unilaterally changing their action (i.e. changing their action whilst the other player does not change their action).

The value of the game corresponding to an equilibrium is the vector of expected payoffs obtained by the players.

Concepts of Solutions to 2-Player Matrix Games - Strong Nash equilibria

A pair of actions (A^*, B^*) is a **strong** Nash equilibrium if

$$R_1(A^*, B^*) > R_1(A, B^*); \quad R_2(A^*, B^*) > R_2(A^*, B).$$

for any action $A \neq A^*$ available to Player 1 and any action $B \neq B^*$ available to Player 2.

i.e. a pair of actions is a strong Nash equilibrium if both players would lose by unilaterally changing their action.

Any Nash equilibrium that is not strong is called weak.

Concepts of Solutions to 2-Player Matrix Games - Mixed Nash equilibria

A pair of mixed strategies, denoted (M_1, M_2) is a mixed Nash equilibrium if

$$R_1(M_1, M_2) \geq R_1(A, M_2); \;\; R_2(M_1, M_2) \geq R_2(M_1, B)$$

for any action (pure strategy) A available to Player 1 and any action B available to Player 2.

It should be noted that the expected reward Player 1 obtains when he plays a mixed strategy M against M_2 is a weighted average of the expected rewards of playing his pure actions against M_2, where the weights correspond to the probability of playing each action.

It follows that player 1 cannot do better against M_2 than by using the best pure action against M_2, i.e. if Player 1 cannot gain by switching to a pure strategy, then she cannot gain by switching to any mixed strategy.

The Bishop-Cannings Theorem

The *support* of a mixed strategy M_1 is the set of actions that are played with a positive probability under M_1.

Suppose (M_1, M_2) is a Nash equilibrium pair of mixed strategies and the support of M_1 is S. We have

$$R_1(A, M_2) = R_1(M_1, M_2), \forall A \in S; \; R_1(B, M_2) < R_1(M_1, M_2), \forall B \notin S.$$

This is intuitively clear, since if a player uses actions A and B under a mixed strategy, then at equilibrium these actions must give the same expected reward, otherwise one action would be preferred over the other.

It follows that at such an equilibrium all actions in the support of M_1 must give the same expected reward, which thus has to be equal to the expected reward of using M_1 (which is calculated as a weighted average). Thus all mixed Nash equilibria are weak.

Nash Equilibria - Results

Every matrix game has at least one Nash equilibrium.

If there is a unique pure Nash equilibrium of a 2×2 game (i.e. a game in which both players have just 2 possible actions), then that is the only Nash equilibrium.

If there are no or two strong Nash equilibria in such a game, then there is always a mixed Nash equilibrium.

Mixed Nash equilibria can be found using the Bishop-Cannings theorem.

Symmetric Games

A game is symmetric if

1. Players all choose from the same set of actions.
2. $R_1(A, B) = R_2(B, A)$.

Note that the symmetric Hawk-Dove game satisfies these conditions.

Symmetric Games

If (A, B) is a pure Nash equilibrium in a symmetric game, then (B, A) is also a pure Nash equilibrium.

At a mixed Nash equilibrium or minimax solution of a symmetric 2×2 game, the players use the same strategy as each other.

Nash Equilibria - Example 3.4.2

Derive all the Nash equilibrium of the following game

	H	D
H	(-2,-3)	(4,0)
D	(0,4)	(3,1)

Nash Equilibria - Example 3.4.2

The pure Nash equilibria can be found by checking every possible pair of actions.

(H, H) is not a Nash equilibrium as either player would prefer to unilaterally change their action to D (and hence obtain 0 rather than -2 or -3).

(D, D) is not a Nash equilibrium as either player would prefer to unilaterally change their action to H (and hence obtain 4).

(H, D) is a Nash equilibrium, since if Player 1 switches to D, she obtains 3 not 4. If Player 2 switches to H, he obtains -3 not 0. The value corresponding to this equilibrium is $(4, 0)$.

Similarly, (D, H) is a Nash equilibrium. The value corresponding to this equilibrium is $(0, 4)$.

Nash Equilibria - Example 3.4.2

To find the mixed Nash equilibrium, we use Bishop-Cannings theorem. Suppose the strategy of Player 1 at equilibrium is $pH + (1-p)D$. Player 2 must be indifferent between his two actions. Hence,

$$R_2(pH + (1-p)D, H) = R_2(pH + (1-p)D, D) = v_2,$$

where v_2 is the value of the game to Player 2 .

We thus have $-3p + 4(1-p) = 0p + 1(1-p) = v_2$. Solving these equations gives $p = 0.5$, $v_2 = 0.5$.

Nash Equilibria - Example 3.4.2

Similarly, suppose the strategy of Player 2 at equilibrium is $qH + (1-q)D$. Player 1 must be indifferent between his two actions. Hence,

$$R_1(H, qH + (1-q)D) = R_1(D, qH + (1-q)D) = v_1.$$

We thus obtain $-2q + 4(1-q) = 0q + 3(1-q) = v_1$. Solving these equations gives $q = \frac{1}{3}$, $v_1 = 2$.

Hence, the mixed equilibrium is $(0.5H + 0.5D, 1/3H + 2/3D)$. The corresponding value is $(2, 0.5)$.

Advantages of the Concept of Minimax Strategies

1. A player only has to know his own payoffs in order to determine his/her minimax strategy and payoff.
2. Apart from degenerate cases (e.g. two actions always give the same payoff), there is a unique minimax strategy.
3. In the case of fixed-sum games, it seems eminently reasonable to follow a minimax strategy.

Disadvantages of the Concept of Minimax Strategies

1. Although the minimax value of a game is well defined, when both players play their minimax strategy their vector of expected payoffs is not necessarily the minimax value of the game (i.e. when both players play the strategy that maximises their guaranteed expected reward, one or more of the players may obtain more than this guaranteed minimum).
2. Many situations cannot be described in terms of pure competition (i.e. in terms of a fixed-sum game). In such cases, the assumption that the aim of an opponent is to minimise a player's expected reward may well be unreasonable.

Advantages of the Concept of Nash Equilibria

1. In fixed sum games, the unique Nash Equilibrium pair of strategies is equal to the pair of minimax strategies.
2. In non-fixed sum games, the Nash equilibrium concept makes the more reasonable assumption that both players wish to maximise their own reward.
3. When using the concept of Nash equilibrium it is normally assumed that players know the payoff functions of their opponents. However, in order to find a pure Nash equilibrium, it is only necessary to be able to order the preferences of opponents, not their actual payoffs.

Disadvantages of the Concept of Nash Equilibria

1. There may be multiple equilibria of a game, so the concept of Nash equilibrium should be strengthened in order to make predictions in such situations.
2. Unlike in the derivation of minimax strategies, it is assumed that the payoff functions of opponents are known. This information is necessary to derive a mixed Nash equilibrium.
3. The concept of a Nash equilibrium requires that a player maximises his/her reward given the behaviour of opponents. However, this behaviour is not known a priori.

3.5 Actions Dominated by Pure Strategies

Suppose that by taking action A_i Player 1 always gets at least the same reward as by playing A_j, regardless of the action taken by Player 2, and for at least one action of Player 2 he obtains a greater reward. Action A_i of Player 1 is said to dominate action A_j.

i.e. action A_i of Player 1 dominates A_j if

$$R_1(A_i, B_k) \geq R_1(A_j, B_k), k = 1, 2, \ldots, n$$

and for some k_0, $R_1(A_i, B_{k_0}) > R_1(A_j, B_{k_0})$.

Actions Dominated by Pure Strategies

Similarly, action B_i of Player 2 dominates action B_j if

$$R_2(A_k, B_i) \geq R_2(A_k, B_j), k = 1, 2, \ldots, m$$

and for some k_0

$$R_2(A_{k_0}, B_i) > R_2(A_{k_0}, B_j).$$

Actions Dominated by Randomised Strategies

When removing dominated strategies, it is easiest to first remove those that are dominated by pure strategies.

Once that is done we then remove those that are dominated by randomised strategies.

The following statements are important in determining which strategies may be dominated by a randomised strategy.

Actions Dominated by Randomised Strategies

If A_i is Player 1's unique best response to B_j, then A_i cannot be dominated by any strategy, pure or randomised.

Thus, if $\exists B_k$ such that $R_1(A_i, B_k) > R_1(A_j, B_k)$, $\forall j \neq i$, then A_i cannot be dominated.

Similarly, if B_i is Player 2's unique best response to A_j, then B_i cannot be dominated.

Actions Dominated by Randomised Strategies

Player 1's mixed strategy $p_1 A_{i_1} + p_2 A_{i_2} + \ldots + p_l A_{i_l}$ dominates A_j, $j \neq i_s$ for any $s = 1, 2, ...l$, if for all B_k

$$R_1(p_1 A_{i_1} + p_2 A_{i_2} + \ldots + p_l A_{i_l}, B_k) \geq R_1(A_j, B_k)$$

and this inequality is strict for at least one value of k.

Similarly, Player 2's mixed strategy $q_1 B_{i_1} + q_2 B_{i_2} + \ldots + q_l B_{i_l}$ dominates B_j, $j \neq i_s$ for any $s = 1, 2, ...l$, if for all A_k

$$R_2(A_k, q_1 B_{i_1} + q_2 B_{i_2} + \ldots + q_l B_{i_l}) \geq R_2(A_k, B_j)$$

and this inequality is strict for at least one value of k.

Successive removal of dominated actions

It is clear that an individual should not use a dominated action. Hence, we may remove such actions from the payoff matrix without changing either the minimax strategies or the Nash equilibria.

It should be noted that an action that was not previously dominated may become dominated after the removal of dominated strategies.

Hence, we continue removing dominated strategies until there are no dominated strategies left in the reduced game (see tutorial and following example).

3.6 Refinements of the concept of Nash equilibrium

We consider 3 refinements of the concept of Nash equilibrium, which are useful when there are multiple equilibria.

1. Subgame perfect Nash equilibria.
2. Payoff dominant Nash equilibria.
3. Risk domninant Nash equilibria.

Subgame Perfect Equilibria in the Context of Matrix Games

We have considered the concept of subgame perfection in extended games.

In the asymmetric Hawk-Dove game, Player 1 will take action H and Player 2 will take action D under the following strategy pairs $(H, [D|H, H|D])$ and $(H, [D|H, D|D])$.

However, when Player 2 plays $[D|H, D|D]$ he does not use his best response when Player 1 makes a "mistake" and plays D. Hence, the second pair of actions does not define a subgame perfect Nash equilibrium.

We will now consider the matrix form of this game (this was derived earlier).

Subgame Perfect Equilibria in the Context of Matrix Games

	$(H\|H, H\|D)$	$(H\|H, D\|D)$	$(D\|H, D\|D)$	$(D\|H, H\|D)$
H	(-2,-2)	(-2,-2)	(4,0)	(4,0)
D	(0,4)	(2,2)	(2,2)	(0,4)

It can be seen that $(H, [D|H, D|D])$ is a (weak) Nash equilibrium in this game, since neither player can increase their payoff by unilaterally switching strategy.

Subgame Perfect Equilibria in the Context of Matrix Games

However, the strategy $[D|H, H|D]$ dominates the action $[D|H, D|D]$, since Player 2 always does as well by playing $[D|H, H|D]$ rather than $[D|H, D|D]$ and does better when Player 1 plays D.

It can be shown that $(H, [D|H, H|D])$ is the only Nash equilibrium left after the removal of dominated strategies.

Payoff Dominant Nash Equilibria

A payoff vector (v_1, v_2) is said to Pareto dominate payoff vector (x_1, x_2) if $v_1 \geq x_1$, $v_2 \geq x_2$ and inequality is strict in at least one of the cases.

That is to say, Nash Equilibrium 1 of a game Pareto dominates Nash Equilibrium 2 if no player prefers Equilibrium 1 to Equilibrium 2 and at least one player prefers Equilibrium 1.

A Nash equilibrium is payoff dominant if the value of the game corresponding to this equilibrium Pareto dominates all the values of the game corresponding to other equilibria.

Example

Consider the following game

	A	B
A	(4,4)	(0,0)
B	(0,0)	(2,2)

Such a game is called a coordination game, as both players would like to take the same action.

Example

There are 3 Nash equilibria

1. (A, A) - Value (4,4).
2. (B, B) - Value (2,2).
3. $(1/3A + 2/3B, 1/3A + 2/3B)$ - Value $(\frac{4}{3}, \frac{4}{3})$.

The first equilibrium Pareto dominates the other two equilibria, whilst the second equilibrium Pareto dominates the third.

(A, A) is the payoff dominant equilibrium.

Risk Dominance

Suppose there are pure Nash equilibria (A, C) and (B, D).

The risk factor associated with strategy A, F_A, is the probability with which Player 2 should play C (which is used at the Nash equilibrium where Player 1 plays A), in order to make Player 1 indifferent between playing A or B.

A high risk factor indicates that Player 1 must be relatively sure that Player 2 will play C for Player 1 to prefer A to B.

Similarly, the risk factor associated with strategy B is the probability with which Player 2 should play D, in order to make Player 1 indifferent between playing A or B.

Risk Dominance

The risk factors associated with C and D can be calculated in a similar way.

The Nash equilibrium (A, C) risk dominates (B, D) if $F_A \leq F_B$, $F_C \leq F_D$ and there is strictly inequality in at least one case.

Example

Consider the following symmetric game

	A	B
A	(4,4)	(-1000,0)
B	(0,-1000)	(2,2)

It is clear that the equilibrium (A, A) payoff dominates the equilibrium (B, B). However, there is a large risk associated with playing action A (the possibility of obtaining a payoff of -1000).

Example

From the symmetry of the game, the risk factors of the two strategies are the same for both players.

The risk factor associated with A is given by the solution of

$$4p - 1000(1-p) = 0p + 2(1-p) \Rightarrow p = \frac{1002}{1006}.$$

The risk factor associated with B is given by the solution to

$$2p + 0(1-p) = -1000p + 4(1-p) \Rightarrow p = \frac{4}{1006}.$$

It follows that B risk dominates A.

Conclusion

If an equilibrium both payoff and risk dominates another, it seems clear that this should be the one chosen.

In other cases, it is not clear what equilibrium should be played.

The concept of risk domination is important in evolutionary game theory (see later).

3.7 2-Player Games with a Continuum of Strategies and Simultaneous Moves

Assume that Player i chooses an action from a finite interval S_i.

The payoff to Player i when Player 1 takes action x_1 and Player 2 takes action x_2 is given by $R_i(x_1, x_2)$.

It is assumed that the payoff functions are differentiable.

The Symmetric Cournot Game

Assume that two firms produce an identical good. Firm i produces x_i units per time interval.

The price of the good is determined by total supply and all production is sold at this "clearing price". It is assumed that $p = A - B[x_1 + x_2]$, $(A, B > 0)$.

The costs of producing x units of the good are assumed to be $C + Dx$ for both firms.

The payoff of a firm is taken to be the profit obtained (revenue minus costs). Revenue is simply production times price.

The Symmetric Cournot Game

The payoff obtained by Firm 1 is given by

$$R_1(x_1, x_2) = px_1 - C - Dx_1 = (A - D)x_1 - Bx_1^2 - Bx_1x_2 - C.$$

By symmetry, the payoff obtained by Player 2 is

$$R_2(x_1, x_2) = px_2 - C - Dx_2 = (A - D)x_2 - Bx_2^2 - Bx_1x_2 - C.$$

It should be noted that it clearly does not pay firms to produce more than the amount x_{max} that guarantees that the price is equal to the unit (marginal) cost of production. Since x_{max} is finite, we may assume that firms choose their strategy (production level) from a finite interval.

Best Response Functions

Given the output of the opponent, we can calculate the optimal response of a player using calculus.

Let $B_1(x_2)$ denote the best response of Player 1 to x_2.

Let $B_2(x_1)$ denote the best response of Player 2 to x_1.

Nash Equilibria in Games with a Continuum of Strategies and Simultaneous Actions

At a Nash equilibrium (x_1^*, x_2^*), we have

$$x_1^* = B_1(x_2^*);\; x_2^* = B_2(x_1^*).$$

Thus, at a Nash equilibrium Player 1 plays her best response to Player 2's strategy and vice versa.

The Cournot Game

Suppose $A = 3$, $B = \frac{1}{1000}$, $C = 100$ and $D = 1$.

We have

$$R_1(x_1, x_2) = 2x_1 - \frac{x_1^2}{1000} - \frac{x_1 x_2}{1000} - 100.$$

In order order to derive the best response of Player 1 to Player 2's action, we differentiate Player 1's payoff function with respect to x_1, his action.

The Cournot Game

We have

$$\frac{\partial R_1(x_1, x_2)}{\partial x_1} = 2 - \frac{2x_1 + x_2}{1000}.$$

It should be noted that this derivative is decreasing in x_1, hence any stationary point must be a maximum.

The optimal response is given by

$$2 - \frac{2x_1 + x_2}{1000} = 0 \Rightarrow x_1 = 1000 - \frac{x_2}{2}.$$

Thus $B_1(x_2) = 1000 - \frac{x_2}{2}$.

It should be noted that this solution is valid as long as $x_2 \leq 2000$ (production cannot be negative).

If $x_2 > 2000$, then $\frac{\partial R_1(x_1,x_2)}{\partial x_1}$ is negative for all non-negative values of x_1.

In this case the best response is not to produce anything.

The Cournot Game

By symmetry the best response of Player 2 is given by

$$B_2(x_1) = \max\{0, 1000 - \frac{x_1}{2}\}.$$

We look for an equilibrium at which both firms are producing. In this case

$$x_1^* = 1000 - \frac{x_2^*}{2};\; x_2^* = 1000 - \frac{x_1^*}{2}.$$

It follows that at Nash equilibrium both firms must produce $\frac{2000}{3}$ units.

Intuitively, from the form of the game both firms should produce the same amount.

The Cournot Game

The value of the game can be found by substituting these values into the payoff functions.

$$R_1(x_1^*, x_2^*) = R_2(x_1^*, x_2^*) = 2 \times \frac{2000}{3} - 2 \times \frac{1000}{9} - 100 \approx 344.$$

The equilibrium price is $3 - 2 \times \frac{2}{3} = \frac{5}{3}$.

The Cournot Game

There cannot be an equilibrium at which one of the firms does not produce. The argument is as follows.

If one firm does not produce, then the optimal response to this is to produce 1000 units.

(0,1000) cannot be a Nash equilibrium, since the best response of Firm 1 to $x_2 = 1000$ is to choose $x_1 = 500$.

The Stackelberg Model

This is identical to the Cournot model, except that it is assumed that one of the firms is a market leader and chooses its production level before the second firm chooses.

The second firm observes the production level of the first.

Games with a Continuum of Strategies and Sequential Moves

Suppose Player 2 moves after Player 1 and observes the action taken by Player 1. The equilibrium is derived by recursion.

Player 2 should choose the optimum action given the action of Player 1.

Hence, we first need to solve

$$\frac{\partial R_2(x_1, x_2)}{\partial x_2} = 0.$$

This gives the optimal response of Player 2 as a function of the strategy of Player 1, $x_2 = B_2(x_1)$.

Games with a Continuum of Strategies and Sequential Moves

We now calculate the optimal strategy of the first player to move.

If Player 1 plays x_1, Player 1 responds by playing $B_2(x_1)$. Hence, we can express the payoff of Player 1 as a function simply of x_1, i.e. $R_1(x_1, B_2(x_1))$.

In order to find the optimal action of Player 1, we differentiate this function with respect to x_1.

Having calculated the optimal value of x_1, we can derive x_2.

Example

Derive the equilibrium of the Stackelberg version of the previous example.

We have

$$
\begin{aligned}
R_2(x_1, x_2) &= 2x_2 - \frac{x_2^2}{1000} - \frac{x_1 x_2}{1000} - 100. \\
\frac{\partial R_2(x_1, x_2)}{\partial x_2} &= 2 - \frac{x_2}{500} - \frac{x_1}{1000}.
\end{aligned}
$$

Example

It follows that the best response of Player 2 is given by

$$2 - \frac{x_2}{500} - \frac{x_1}{1000} = 0 \Rightarrow x_2 = 1000 - \frac{x_1}{2}.$$

Hence,

$$\begin{aligned} R_1(x_1, B_2(x_1)) &= R_1(x_1, 1000 - \frac{x_1}{2}) \\ &= 2x_1 - \frac{x_1^2}{1000} - \frac{x_1(1000 - x_1/2)}{1000} - 100 \\ &= x_1 - \frac{x_1^2}{2000} - 100. \end{aligned}$$

Example

Differentiating

$$\frac{\partial R_1(x_1, B_2(x_1))}{\partial x_1} = 1 - \frac{x_1}{1000}.$$

It follows that Firm 1 maximises its profit by producing 1000 units.

The best response of Firm 2 is $B_2(x_1) = 1000 - \frac{x_1}{2} = 500$ units.

The Stackelberg equilibrium is $(1000, 500)$. Hence, the leader produces more than at the Cournot equilibrium and the follower produces less.

Total production is greater than at the Cournot equilibrium, i.e. the equilibrium price is lower.

Example

The profit of Firm 1 at this equilibrium is

$$R_1(1000, 500) = 2 \times 1000 - 1000 - 500 - 100 = 400.$$

The profit of Firm 2 at this equilibrium

$$R_2(1000, 500) = 2 \times 500 - 250 - 500 - 100 = 150.$$

It is clear that Firm 1 gains by being the leader. Firm 2 loses. The sum of profits is lower than at the Cournot equilibrium.

This seems somewhat counter-intuitive, as the market would seem to be more competitive under the Cournot model.