**Dynamical Systems and Mathematical Models**

A system is any collection of objects and their interconnections that we happen to be interested in; be it physical, engineering, economic, financial, demographic, social or whatever. A *dynamical system* is one which evolves with time.
A mathematical model of such a system is a collection of quantities (variables and parameters) and their interrelationships (usually equations) which describe how the system behaves. More than that, it is argued that a “useful” mathematical model is one which not only explains the system’s behaviour to date but can also be used to predict the future behaviour of the system; the ability to predict a system’s behaviour is the first step in controlling it. We’ll tacitly assume that having identified those aspects of a system that we wish to model, it is possible to build a mathematical model, and that having done so, we’ll use the terms system and mathematical model loosely and interchangeably.

What’s the difference between a variable and a parameter ? The answer is essentially one of semantics, but how these words are used throughout this module is best illustrated by means of an example : Consider modelling the passage of a small ball falling through the air towards the ground under the influence of gravity and wind fluctuations. Time is obviously an important variable as are the position of the ball as a function of time and the strength and direction of the wind as functions of time. The mass of the ball, the gravitational “constant” and any frictional “constants” are usually considered as parameters of this system. Thus variables are quantities which vary significantly during the time period over which the system is observed, while parameters are quantities which are constant or whose variation is small and irrelevant in comparison with that of the variables ( which makes it attractive to model this type of parameter as a constant) over the same time period.

Depending on how frequently the system is observed, the associated mathematical model may be *continuous-time*: all variables are continuously monitored - thus the “lifetime” for which the model is valid is an interval; or *discrete-time*: all variables are monitored at discrete points in time - and so the time period of interest is a countable set. We shall use $t_0$ and $t_f$ as the initial and final times of the relevant time period. For continuous-time models, we shall use $t$ to represent an arbitrary time instant in $[t_0, t_f]$. For discrete-time models, $t_k,\ k = 0, 1, 2, \ldots$ will represent the $k$-th observation time. In addition, we’ll usually restrict ourselves to *equally spaced* observations. In which case if $M + 1$ observation times are available to us between $t_0$ and $t_f$, then with $\Delta = (t_f - t_0)/M$ we have

$$t_k = t_0 + k\Delta.$$

For a given system and time period, $t_0$ and $\Delta$ are fixed and so $t_k$ and $k$ are equivalent, and for convenience we'll use $k$ as the time variable.

Consider the falling ball example again. In a situation where the wind fluctuations are negligible, it is in principle possible to compute exactly the position of the ball as a function of time given that we know the parameter values and the initial position and velocity of the ball - by integrating the differential equation given by Newton's 2nd Law. Indeed, in a windy situation, if we know the wind fluctuations (i.e. have a function for them), we could solve the differential equation to get the future position of the ball exactly. In practice, however, we don't know in advance how the wind will fluctuate, and thus it is impossible to compute exactly the future position of the ball (we might tackle this situation using probabilistic or statistical methods).

Such examples lead us to the following classification. A system is called *deterministic* if knowledge of its initial variable values, its parameter values and all input functions over time is sufficient to compute *exactly* its future behaviour ( computing "future behaviour" means computing future values of all relevant variables). Systems for which we can only estimate or predict future behaviour in some statistical sense are called *stochastic*. It is common practice to approximate a stochastic system by a simplified deterministic model usually because it is argued that such a model captures the essential features of the system. All the models dealt with in this module are deterministic.

# 1 State Models

Phrases such as "state of the country", "state of the economy", "state of the weather" conjure up somewhat vague images of how the complex system under consideration is currently behaving. The mathematical model known as the state model attempts to reflect this notion, yet being mathematical it is precisely defined and thus perhaps more limited. It is closely related to and developed out of the notion of a deterministic system, though it is easily adapted to describe stochastic systems.

If we place ourselves in the position of an external observer, then once we identify the system of interest, we can distinguish between it and its environment. (The border between the two can in reality be fuzzy with the result that the border between the mathematical model and its environment can appear arbitrary). We place the variables associated with the mathematical model in three categories:

- Input variables or inputs - those that describe how the environment influences the system, or those that the observer can 'choose' to affect the behaviour of the system. We'll use $u_1(t), u_2(t), \ldots, u_r(t)$ to denote the values of the inputs to a continuous-time model with $r$- inputs. As a shorthand, we'll use vector notation as follows

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{pmatrix}$$

  For a discrete-time model with equally spaced observations, we'll similarly use $\mathbf{u}(k)$. It is not necessary for a dynamic system to have inputs. Such systems are called unforced or *autonomous*.

- Output variables or outputs - those that describe how the system influences the environment, or the variables that we observe. We'll denote the outputs of a $m$-output system by $y_1(t), y_2(t), \ldots, y_m(t)$ Again we'll use

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{pmatrix}$$

  and similarly $\mathbf{y}(k)$. In many situations, the outputs are a subset of the state variables.

- State variables - the current state of the system is described by the values of a set of variables (the state variables or internal variables), which contain sufficient information to compute exactly the future states of the system (if the future inputs are known). The number of state variables is called the order of the system. It is an important measure of system complexity. We'll denote it by $n$. Then we'll denote the state variables by $x_1(t), x_2(t), \ldots, x_n(t)$ and the state vector by

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

or for discrete-time models by $\mathbf{x}(k)$. We sometimes refer to the state vector as "the state", and to a particular value of the state vector as " a state".

For many high order systems, it is usual to have

$$n \gg r \qquad\qquad n \gg m$$

We next look at the relationships between variables. Implicit in the definition of the state is that future states can be computed from the current state and knowledge of any inputs now and to come. This leads us to a functional relationship called the state equation. The state equation is a "dynamic" equation in that its solution tells us how the state evolves with time, but the exact nature of this equation depends on whether we are dealing with a discrete-time or continuous-time model.

## 1.1 State Equation for discrete-time Models

The state equation details the relationship between the next state and the current state, current input if any, and current time if appropriate. Thus for systems with equally spaced observations, the most general form is

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k), k)$$

where $\mathbf{f}$ is an appropriate function. Such an model is said to be *time- varying.* If there is no explicit dependence on time, then the model is said to be *time invariant.* So for instance a time-invariant system with no input has a state equation of form

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$$

Many of the examples we will meet are of the form

$$\mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) = \mathbf{F}(\mathbf{x}(k)) + \mathbf{g}(\mathbf{x}(k))\mathbf{u}(k)$$

This is referred as an affine control system. A particularly common type of affine system is the linear system

$$\mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) = A\mathbf{x}(k) + B\mathbf{u}(k)$$

## 1.2 State Equation for continuous-time Models

There is no such thing as the "next" time and hence no "next" state. However if the state is differentiable, then the rate of change of the state with respect to time will only depend on the current state, current input if any, and current time if appropriate, and so the state may be represented as the solution of the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

where $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$ and the function $\mathbf{f}$ is such that this equation has a unique solution in $(t_0, t_f)$ for each initial state $\mathbf{x}(t_0)$ and all forcing functions $\mathbf{u}(t)$. Again, this formulation of the model is said to be time-varying. We can also have time-invariant models, and so for example a time-invariant system with no input has a state equation of form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$$

Again many of the examples will be affine or linear.

## 1.3 Output Equation

In addition to the state equation, the state model requires an output equation which tells us how the current outputs depend on the current state, (occasionally the current inputs) and current time. For both discrete-time and continuous-time systems, the form of this "static" equation is therefore

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t)$$

where $\mathbf{h}$ is an appropriate function.
For many low order systems, it is usual to have $\mathbf{y}(t) = \mathbf{x}(t)$.