Some Notes on Differential Geometry

We will start with some terminology from Differential Geometry. Let

$$h : \mathbb{R}^n \to \mathbb{R}$$

be a C^1 (or smooth) *function* defined on the region $\mathcal{N}$ and

$$\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$$

be C^1 *vector fields* defined on the region $\mathcal{N}$. The *gradient* of h is the $1 \times n$ row vector defined by

$$\frac{\partial h}{\partial \mathbf{x}} \triangleq \begin{pmatrix} \frac{\partial h}{\partial x1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \end{pmatrix}$$

while the *Jacobian* of the vector field f is the $n \times n$ matrix defined by

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \triangleq \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Lie Derivative The *Lie* derivative of h with respect to f is defined by

$$L_{\mathbf{f}} h \triangleq \frac{\partial h}{\partial \mathbf{x}} \mathbf{f}$$

i.e. the directional derivative of h in the direction f. Note that $L_{\mathbf{f}} h$ is a *function* in its own right. Furthermore we will make use of the following recursive definitions

$$\begin{aligned} L_{\mathbf{f}^0} h &= h \\ L_{\mathbf{f}^{k+1}} h &= L_{\mathbf{f}} L_{\mathbf{f}^k} h \\ L_{\mathbf{g}} L_{\mathbf{f}} h &= L_{\mathbf{g}} \left(L_{\mathbf{f}} h \right) \end{aligned}$$

Lie Bracket The *Lie* Bracket of the vector fields f and g is defined by

$$[f,\ g] \triangleq \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{g}$$

Note that $[f,\ g]$ is a vector field in its own right. Furthermore we will use the notation

$$ad_{\mathbf{f}} \mathbf{g} \triangleq [f,\ g]$$

(*ad* stands for adjoint) and on which the following recursive definitions are based

$$
\begin{aligned}
ad_{\mathbf{f}^0}\mathbf{g} &= \mathbf{g} \\
ad_{\mathbf{f}^{k+1}}\mathbf{g} &= [f,\ ad_{\mathbf{f}^k}\mathbf{g}]
\end{aligned}
$$

The following properties of the *Lie* Bracket are useful for computational purposes

1. Bilinearity: For the constant scalars α_1 and α_2

$$[\alpha_1\mathbf{f}_1 + \alpha_2\mathbf{f}_2,\ \mathbf{g}] = \alpha_1[\mathbf{f}_1,\ \mathbf{g}] + \alpha_2[\mathbf{f}_1,\ \mathbf{g}]$$

$$[\mathbf{f},\ \alpha_1\mathbf{g}_1 + \alpha_2\mathbf{g}_2] = \alpha_1[\mathbf{f},\ \mathbf{g}_1] + \alpha_2[\mathbf{f},\ \mathbf{g}_2]$$

2. Skew-commutativity

$$[\mathbf{f},\ \mathbf{g}] = -[\mathbf{g},\ \mathbf{f}]$$

3. *Jacobi* Identity

$$L_{ad_{\mathbf{f}}\mathbf{g}}h = L_{\mathbf{f}}L_{\mathbf{g}}h - L_{\mathbf{g}}L_{\mathbf{f}}h$$

Examples:

$$h = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2, \qquad \mathbf{f} = \begin{pmatrix} x_1x_2 \\ \sin x_1 \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\partial h}{\partial \mathbf{x}} = \begin{pmatrix} x_1, & x_2 \end{pmatrix}, \qquad \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} x_2 & x_1 \\ \cos x_1 & 0 \end{pmatrix}, \qquad \frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{\mathbf{f}}h = \frac{\partial h}{\partial \mathbf{x}}\mathbf{f} = \begin{pmatrix} x_1, & x_2 \end{pmatrix}\begin{pmatrix} x_1x_2 \\ \sin x_1 \end{pmatrix} = x_1^2x_2 + x_2\sin x_1$$

$$L_{\mathbf{g}}L_{\mathbf{f}}h = \left(\frac{\partial L_{\mathbf{f}}h}{\partial \mathbf{x}}\right)\mathbf{g} = \begin{pmatrix} 2x_1x_2 + x_2\cos x_1, & x_1^2 + \sin x_1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_1^2 + \sin x_1$$

$$[\mathbf{f},\ \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1x_2 \\ \sin x_1 \end{pmatrix} - \begin{pmatrix} x_2 & x_1 \\ \cos x_1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}$$

$$ad_{\mathbf{f}}\mathbf{g} = [\mathbf{f},\ \mathbf{g}] = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
ad_{\mathbf{f}^2}\mathbf{g} = [\mathbf{f},\ ad_{\mathbf{f}}\mathbf{g}] &= \left(\frac{\partial ad_{\mathbf{f}}\mathbf{g}}{\partial \mathbf{x}}\right)\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(ad_{\mathbf{f}}\mathbf{g}) \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1x_2 \\ \sin x_1 \end{pmatrix} - \begin{pmatrix} x_2 & x_1 \\ \cos x_1 & 0 \end{pmatrix}\begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1\cos x_1 \end{pmatrix}
\end{aligned}
$$

A technical **Lemma**: Let h be C^1 on $\mathcal{N}$. Then

$$L_{\mathbf{g}}h = L_{\mathbf{g}}L_{\mathbf{f}}h = \cdots = L_{\mathbf{g}}L_{\mathbf{f}^k}h = 0$$

is equivalent to

$$L_{\mathbf{g}}h = L_{ad_{\mathbf{f}}\mathbf{g}}h = \cdots = L_{ad_{\mathbf{f}^k}\mathbf{g}}h = 0$$

Proof: (exercise - by induction on k and using the *Jacobi* identity).

The Frobenius Theorem Is there a solution (h) to the set of linear pdes e.g.

$$\begin{aligned} \frac{\partial h}{\partial \mathbf{x}}\mathbf{f}_1 &= 0 \\ \frac{\partial h}{\partial \mathbf{x}}\mathbf{f}_2 &= 0 \end{aligned}$$

where $\mathbf{f}_i,\ i = 1, 2$ are known vector fields? Clearly the solution, if it exists, depends entirely on the $\mathbf{f}_i$. If there is a solution we say that the set $\{\mathbf{f}_i\}$ is *completely integrable.*
It transpires that a solution exists if and only if there exist scalars $\alpha_i(\mathbf{x}),\ i = 1, 2$ such that

$$[\mathbf{f}_1,\ \mathbf{f}_2] = \alpha_1\mathbf{f}_1 + \alpha_2\mathbf{f}_2$$

i.e. the *Lie* bracket of $\mathbf{f}_1$ and $\mathbf{f}_2$ can be expressed as a linear combination of $\mathbf{f}_1$ and $\mathbf{f}_2$ themselves. This condition is known the *involutivity condition* on the vector fields. (In 3-d for instance, it says that the *Lie* bracket of the vector fields must lie in the plane that is formed by the vector fields themselves). The involutivity condition is easy to check, and so we can easily determine if the system of pdes is solvable.
This is the *Frobenius* theorem in the case of 2 vector fields in 3-d space.
More generally, for the case of m linearly independent vector fields in n - space, the set $F_m = \{\mathbf{f}_1,\ \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is said to be completely integrable if there exits solutions $h_1(\mathbf{x}),\ h_2(\mathbf{x}), \ldots, h_{n-m}(\mathbf{x})$ to the set of equations

$$\frac{\partial h_i}{\partial \mathbf{x}}\mathbf{f}_j = 0, \qquad i = 1, 2, \ldots, n-m,\ \ j = 1, 2, \ldots, m$$

and the gradients $\frac{\partial h_i}{\partial \mathbf{x}}$ are linearly independent.
In addition the set F_m is said to be involutive if there exist scalars $\alpha_{ijk}(\mathbf{x})$ such that

$$[\mathbf{f}_i,\ \mathbf{f}_j] = \sum_{k=1}^{m} \alpha_{ijk}\mathbf{f}_k, \qquad 1 \leq i, j \leq m$$

Theorem(*Frobenius*): The linearly independent set $\{\mathbf{f}_1,\ \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is completely integrable if and only if it is involutive.

Example: Does there exist a solution of

$$\begin{aligned} 2x_2\frac{\partial h}{\partial x_1} - 3x_1\frac{\partial h}{\partial x_3} &= 0 \\ x_1\frac{\partial h}{\partial x_1} + 2x_2\frac{\partial h}{\partial x_2} + 2\frac{\partial h}{\partial x_3} &= 0\ ? \end{aligned}$$

The vector fields are

$$\mathbf{f}_1 = \begin{pmatrix} 2x_2 \\ 0 \\ -3x_1 \end{pmatrix}, \qquad \mathbf{f}_2 = \begin{pmatrix} x_1 \\ 2x_2 \\ 2 \end{pmatrix}$$

which are linearly independent by inspection.

$$\begin{aligned} [\mathbf{f}_1,\ \mathbf{f}_2] = \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}}\mathbf{f}_1 - \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}}\mathbf{f}_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2x_2 \\ 0 \\ -3x_1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ 2x_2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ 0 \\ 3x_1 \end{pmatrix} \\ &= -1\begin{pmatrix} 2x_2 \\ 0 \\ -3x_1 \end{pmatrix} + 0\begin{pmatrix} x_1 \\ 2x_2 \\ 2 \end{pmatrix} = -\mathbf{f}_1 + 0\mathbf{f}_2 \end{aligned}$$

The set $\{\mathbf{f}_1,\ \mathbf{f}_2\}$ is involutive and so there exists a solution to the set of pdes given above.

State Transformations Consider the affine system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{1}$$

where we assume that this system has a fixed point $\mathbf{x}_e = \mathbf{0}$ corresponding to $u \equiv 0$. $\mathcal{N}$ represents a neighbourhood of this fixed point. The mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism if it is C^1 (smooth) on $\mathcal{N}$ and Φ^{-1} exists and is smooth.

System (1) is said to be *input-state linearisable* if there exists a diffeomorphism Φ and a *linearising control law* $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v$ such that the *linearising state* $\mathbf{z} = \Phi(\mathbf{x})$ and new input v satisfy

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\mathbf{z} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v \tag{2}$$

Theorem System (1) is input-state linearisable if and only if there exists $\mathcal{N}$ such that

- the set $F_n = \{\mathbf{g},\ ad_{\mathbf{f}}\mathbf{g}, ad_{\mathbf{f}^2}\mathbf{g}, \ldots, ad_{\mathbf{f}^{n-1}}\mathbf{g}\}$ is linearly independent in $\mathcal{N}$.
- the set $F_{n-1} = \{\mathbf{g},\ ad_{\mathbf{f}}\mathbf{g}, ad_{\mathbf{f}^2}\mathbf{g}, \ldots, ad_{\mathbf{f}^{n-2}}\mathbf{g}\}$ is involutive in $\mathcal{N}$.

Proof:

(1) (Necessity): Assume Φ exists such that system (1) can be transformed into (2). Then

$$\begin{aligned} \mathbf{z} &= \Phi(\mathbf{x}) \\ \Rightarrow \dot{\mathbf{z}} &= \frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{f} + \mathbf{g}u) \end{aligned}$$

In terms of the i-th component of $\mathbf{z}$, this becomes

$$\begin{aligned} \dot{z}_i &= \frac{\partial \Phi_i}{\partial \mathbf{x}}(\mathbf{f} + \mathbf{g}u) \\ &= \frac{\partial z_i}{\partial \mathbf{x}} + \frac{\partial z_i}{\partial \mathbf{x}}\mathbf{g}u \\ &= L_{\mathbf{f}} z_i + L_{\mathbf{g}} z_i u, \qquad i = 1, 2, \ldots n \end{aligned} \tag{3}$$

Comparing the right hand sides of (3) and (2) gives

(i)

$$L_{\mathbf{f}} z_i = z_{i+1}, \qquad i = 1, 2, \ldots, n-1$$

which can be solved inductively to give

$$z_i = L_{\mathbf{f}^{i-1}} z_1, \qquad i = 1, 2, \ldots, n \tag{4}$$

and

(ii)

$$L_{\mathbf{g}} z_i = 0, \qquad i = 1, 2, \ldots, n-1 \tag{5}$$

$$L_{\mathbf{g}} z_n \neq 0 \tag{6}$$

Using (4), (5) and (6) become

$$L_{\mathbf{g}} L_{\mathbf{f}^{i-1}} z_1 = 0, \qquad i = 1, 2, \ldots, n-1 \tag{7}$$

$$L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} z_1 \neq 0$$

respectively, and using the technical Lemma, these in turn become

$$L_{ad_{\mathbf{f}^{i-1}}\mathbf{g}} z_1 = 0, \qquad i = 1, 2, \ldots, n-1 \tag{8}$$

$$L_{ad_{\mathbf{f}^{n-1}}\mathbf{g}} z_1 \neq 0 \tag{9}$$

Now, if F_n is not linearly independent, there exists k such that

$$\begin{aligned} ad_{\mathbf{f}^k}\mathbf{g} &= \sum_{i=1}^{k} \gamma_i ad_{\mathbf{f}^{i-1}}\mathbf{g} \\ \Rightarrow \text{ (by induction)} \qquad ad_{\mathbf{f}^{n-1}}\mathbf{g} &= \sum_{i=1^*}^{n-1} \gamma_i ad_{\mathbf{f}^{i-1}}\mathbf{g} \end{aligned}$$

from which we get

$$\begin{aligned} L_{ad_{\mathbf{f}^{n-1}}\mathbf{g}} z_1 &= \sum_{i=1^*}^{n-1} \gamma_i L_{ad_{\mathbf{f}^{i-1}}\mathbf{g}} z_1 \\ &= 0, \qquad \text{using (8)} \end{aligned}$$

which contradicts (9). Thus F_n is linearly independent.

Since there exists z_1 which satisfies (8), i.e. the vector fields of F_{n-1} are completely integrable and hence by the *Frobenius* theorem, F_{n-1} is involutive.

(2) (Sufficiency): From the involutivity of F_{n-1}, there exists a non-zero function z_1 such that (8) holds. Using the technical Lemma, this implies that (7) holds.
If we choose as the transformed state $\mathbf{z} = \left(z_1, \quad L_{\mathbf{f}} z_1, \quad \ldots \quad L_{\mathbf{f}^{n-1}} z_1\right)^T$, then

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \qquad i = 1, 2, \ldots, n-1 \\ \dot{z}_n &= L_{\mathbf{f}^n} z_1 + L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} z_1 u \end{aligned} \tag{10}$$

To verify that the nonlinearity in the state equation for z_n can be cancelled in order to arrive at (2), it is necessary to show that the coefficient of u in (10) is not zero. From the technical Lemma,

$$L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} z_1 \neq 0 \Leftrightarrow L_{ad_{f^{n-1}}\mathbf{g}} z_1 \neq 0$$

but $L_{ad_{f^{n-1}}\mathbf{g}} z_1 \neq 0$ for otherwise

$$\frac{\partial z_1}{\partial \mathbf{x}} \left(\mathbf{g}, \quad ad_{\mathbf{f}}\mathbf{g}, \quad \ldots, \quad ad_{f^{n-1}}\mathbf{g}\right) = 0$$

would imply that F_n is not linearly independent.