Controllability

In everyday language, a system is controllable if an input exists which is capable of driving the system from an arbitrary initial state to an arbitrary final state.
Mathematically, some care needs to be exercised to take account of admissible control values, distinguishing between controllable and uncontrollable states, and continuous-time and discrete-time systems. From when the idea of controllability was first introduced (c 1960), there have been many revisions to the formal definitions and many related concepts defined e.g. null controllability, reachability, attainability and accessibility.

1 Continuous-time Systems

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \mathbf{u}(.) \in \Omega \subseteq \mathbf{R}^r \tag{1}$$

where Ω is the input constraint set or set of admissible controls (e.g. $|u(t)| \leq 1$) and r is the number of input variables. If the input is unbounded, we shall omit all reference to Ω.

DEFINITION 1: The state $\mathbf{x}_0$ is *null controllable* if there exists an admissible control $\mathbf{u}(t),\ t_0 \leq t < t_f$ such that the corresponding trajectory $\mathbf{x}(t)$ of EQ(1) with $\mathbf{x}(t_0) = \mathbf{x}_0$ has $\mathbf{x}(t_f) = \mathbf{0}$.

DEFINITION 1A (NC): The system defined by EQ(1) is null controllable if every $\mathbf{x}_0$ is null controllable.

DEFINITION 2: The state $\mathbf{x}_f$ is *reachable* if there exists an admissible control $\mathbf{u}(t),\ t_0 \leq t < t_f$ such that the corresponding trajectory $\mathbf{x}(t)$ of EQ(1) with $\mathbf{x}(t_0) = \mathbf{0}$ has $\mathbf{x}(t_f) = \mathbf{x}_f$.

DEFINITION 2A (R): The system defined by EQ(1) is reachable if every $\mathbf{x}_f$ is reachable.

DEFINITION 3 (CC): The system of EQ(1) is *completely controllable* if for each $\mathbf{x}_0$ and $\mathbf{x}_f$, there exists an admissible control $\mathbf{u}(t),\ t_0 \leq t < t_f$ such that the corresponding trajectory $\mathbf{x}(t)$ of EQ(1) with $\mathbf{x}(t_0) = \mathbf{x}_0$ has $\mathbf{x}(t_f) = \mathbf{x}_f$.

From the definitions we see that
CC $\Rightarrow$ NC
CC $\Rightarrow$ R
NC & R $\Rightarrow$ CC

Example:
Consider the system

$$\dot{x} = -2x + u$$

with u unbounded. Show that the control $u(t) = e^{-2t}\left(\frac{x_f e^{2t_f} - x_0}{t_f}\right),\ 0 \leq t < t_f$ transfers x_0 to x_f. Hence this system is CC.

Consider the same system but this time with $|u(t)| \leq 1$. The system is NC (For instance, apply $u(t) \equiv 0$ for sufficient time (t_s) so that $|x(t_s)| < 1/2$; then set $u(t) = 2x - \epsilon\frac{x}{|x|},\ t_s \leq t < t_s + \frac{|x(t_s)|}{\epsilon}$ where $0 < \epsilon < 1 - 2|x(t_s)|$). However it is not R: show that no x_f with $|x_f| > 1/2$ can be reached.

It is possible to extend the equivalences between the definitions for

1.1 Linear Time Invariant (LTI) Systems with Unbounded Controls

For the LTI system

$$\dot{\mathbf{x}} = A\,\mathbf{x} + B\,\mathbf{u} \tag{2}$$

with unbounded controls, we have
NC $\Rightarrow$ CC
R $\Rightarrow$ CC

The LTI system of EQ(2) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ has solution

$$\mathbf{x}(t) = e^{A(t-t_0)}\left(\mathbf{x}_0 + \int_{t_0}^{t} e^{-A\tau}B\mathbf{u}(\tau)d\tau\right)$$

or, equivalently,[1] with $t = t_f$ and $\mathbf{x}(t) = \mathbf{x}_f$

$$\mathbf{0} = -e^{A(t_0-t_f)}\mathbf{x}_f + \mathbf{x}_0 + \int_{t_0}^{t} e^{-A\tau}B\mathbf{u}(\tau)d\tau \tag{3}$$

If the system is NC, then for any $\tilde{\mathbf{x}}_0$ we can always find $\tilde{\mathbf{u}}(t),\ t_0 \leq t < t_f$ so that the equation

$$\mathbf{0} = \tilde{\mathbf{x}}_0 + \int_{t_0}^{t_f} e^{-A\tau}B\tilde{\mathbf{u}}(\tau)d\tau \tag{4}$$

is satisfied. In particular, if $\tilde{\mathbf{x}}_0 = -e^{A(t_0-t_f)}\mathbf{x}_f + \mathbf{x}_0$, we have that this control transfers (arbitrary) $\mathbf{x}_0$ to (arbitrary) $\mathbf{x}_f$.

If the system is R, then for any $\hat{\mathbf{x}}_f$ we can always find $\hat{\mathbf{u}}(t),\ t_0 \leq t < t_f$ so that the equation

$$\mathbf{0} = -e^{A(t_0-t_f)}\hat{\mathbf{x}}_f + \int_{t_0}^{t} e^{-A\tau}B\hat{\mathbf{u}}(\tau)d\tau$$

is satisfied. In particular, if

$$-e^{A(t_0-t_f)}\hat{\mathbf{x}}_f = -e^{A(t_0-t_f)}\mathbf{x}_f + \mathbf{x}_0 \quad \Rightarrow \quad \hat{\mathbf{x}}_f = \mathbf{x}_f - e^{A(t_f-t_0)}\mathbf{x}_0$$

we have that this control transfers (arbitrary) $\mathbf{x}_0$ to (arbitrary) $\mathbf{x}_f$.

The time interval $[t_0, t_f]$ over which the control acts is, for LTI systems with unbounded controls, of little importance in this context: Show that if $\mathbf{u}(t), t_0 \leq t < t_f$ drives $\mathbf{x}_0$ to $\mathbf{x}_f$, then so does $\tilde{\mathbf{u}}(t), \tilde{t}_0 \leq t < \tilde{t}_0 + \frac{t_f-t_0}{s}$ where $\tilde{\mathbf{u}}(t) = \mathbf{u}(t_0 + s(t - \tilde{t}_0))$. ($s$ is a scaling factor.)

[1] where we have used the invertibility of the state transition matrix

1.2 LTI Systems with unbounded controls: Controllability Criteria

1.2.1 Modal Criterion

If the state matrix A has distinct eigenvalues and so is diagonalisable, then consider the modal transformation $\mathbf{x} = E\mathbf{z}$ which converts EQ(2) to

$$\dot{\mathbf{z}} = \Lambda\,\mathbf{z} + \tilde{B}\mathbf{u} \tag{5}$$

where $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Eq(5) may be rewritten

$$\dot{z}_j = \lambda_j z_j + \tilde{B}_j\mathbf{u} \qquad j = 1, 2, \ldots, n$$

where $\tilde{B}_j$ is the j-th row of $\tilde{B}$.
Thus, a necessary condition for $\mathbf{u}$ to influence the mode z_j is that $\tilde{B}_j \neq \mathbf{0}$, or, more generally, a necessary condition for EQ(5) to be CC is that no row of $\tilde{B}$ be zero. This is also sufficient. This leads to

MODAL CRITERION: EQ(5) is CC if and only if no row of $\tilde{B}$ is zero.

We can categorise the j-th mode of a system as controllable or uncontrollable depending on whether it can be controlled by $\mathbf{u}$. The system will be practically useless if any uncontrollable modes are unstable.
One of the main themes running throughout this course is stabilisability.
DEFINITION 4: A system is *stabilisable* if there exists an admissible control which drives any initial state to equilibrium (at $\mathbf{x}_e = \mathbf{0}$), at least asymptotically. To highlight the connection between uncontrollable modes and stabilisability, an equivalent definition is sometimes given

DEFINITION 4A: The system of EQ(2) is *stabilisable* if all its uncontrollable modes are asymptotically stable.

1.2.2 Rank Criterion

A consequence of the *Cayley-Hamilton* theorem is that

$$e^{At} = \sum_{i=0}^{n-1} s_i(t)A^i$$

where the $s_i(t)$ are real scalars. Substituting this into EQ(4) gives

$$\begin{aligned}
\mathbf{0} &= \tilde{\mathbf{x}}_0 + \int_{t_0}^{t_f}\sum_{i=0}^{n-1} s_i(-\tau)A^iB\mathbf{u}(\tau)d\tau \\
&= \tilde{\mathbf{x}}_0 + \sum_{i=0}^{n-1} A^iB\int_{t_0}^{t_f} s_i(-\tau)\mathbf{u}(\tau)d\tau \\
&= \tilde{\mathbf{x}}_0 + \left[B, AB, A^2B, \ldots, A^{n-1}B\right]\mathbf{v} \qquad (6)
\end{aligned}$$

where $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1})^T$ and $\mathbf{v}_i = \int_{t_0}^{t_f} s_i(-\tau)\mathbf{u}(\tau)d\tau$.

For NC (and hence CC), given an arbitrary $\tilde{\mathbf{x}}_0$ EQ(6) must be solvable for $\mathbf{v}$. This only occurs when the *controllability matrix*

$$U \stackrel{\text{def}}{=} \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

has full rank. Since U is $n \times nr$, this leads to

RANK CRITERION: The system of EQ(2) is CC if and only if $\text{rank}(U) = n$.

We also say that the pair (A, B) is CC whenever U has full rank.

If $\text{rank}(U) < n$, then not all states are controllable. In fact, inspecting EQ(6) shows that any NC state $\tilde{\mathbf{x}}_0$ must be in the column space of U, i.e. must be expressible as a linear combination of the columns of U. Thus we can find controls that will drive the system between any two such states.

1.2.3 *Popov-Belevitch-Hautus* (PBH) Test for controllability

Consider the $n \times (n + r)$ matrix

$$M \triangleq [\lambda I - A,\ B], \qquad \text{for all } \lambda \in \mathbf{C}. \tag{7}$$

TEST: The system of Eqn (2) is CC if and only if

$$\text{rank } M = n.$$

We note that if λ is not an eigenvalue of A, then $\lambda I - A$ has rank n and so the test is automatically satisfied. Furthermore this means we only need to check the rank of M at the eigenvalues of A. If the test fails at a particular value of λ, then the mode corresponding to that eigenvalue is uncontrollable.

In addition we can adapt the PBH Test to check for stabilisability. Recall that a system is stabilisable if and only if its unstable modes are controllable. Thus by performing the PBH test for the unstable eigenvalues of A will establish whether the system is stabilisable or not.

1.3 LTI systems: Controllable Canonical Form (CCF)

There are several canonical forms associated with controllability. We shall look at one which is used in several design algorithms, but we shall restrict our attention to single input systems.

Consider an arbitrary single input system

$$\dot{\mathbf{x}} = A\,\mathbf{x} + b\,\mathbf{u} \tag{8}$$

with controllability matrix

$$U = \left[b, Ab, A^2b, \ldots, A^{n-1}b\right]$$

The single input system

$$\dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{b}\hat{\mathbf{u}} \tag{9}$$

where

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \qquad \hat{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{10}$$

has controllability matrix

$$\hat{U} = \left[\hat{b}, \hat{A}\hat{b}, \hat{A}^2\hat{b}, \ldots, \hat{A}^{n-1}\hat{b}\right] = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & - \\ \vdots & \vdots & & - & - \\ 0 & 1 & \cdots & - & - \\ 1 & - & \cdots & - & - \end{pmatrix}$$

Where the $-$ indicate an irrelevant entry. $\hat{A}$ is alternatively referred to as a *companion form* or *Frobenius* matrix,[2] while the system described by EQ(9) is said to be in *controllable canonical form* (CCF).

The rank of $(\hat{U})$ is n and so the system is CC.

We can transform the arbitrary system EQ(8) into CCF if and only if the pair (A, b) is CC. More specifically, this is achieved by the transformation $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ where $\hat{T} = U\hat{U}^{-1}$. Exercise: Prove this result.

2 Discrete-Time Systems

Many of the definitions and concepts associated with continuous-time systems carry over to discrete-time with only the obvious modifications.
For the system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k) \qquad \mathbf{u}_k \in \Omega \subseteq \mathbf{R}^r \tag{11}$$

the definition of a null controllable state reads
DEFINITION 1: The state $\tilde{\mathbf{x}}_0$ is null controllable if there exists an admissible control (a sequence of length N) $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{N-1}$ such that the corresponding trajectory $\mathbf{x}_k$ of EQ(11) with $\mathbf{x}_0 = \tilde{\mathbf{x}}_0$ has $\mathbf{x}_N = \mathbf{0}$.

Definitions 1A, 2, 2A and 3 are modified accordingly. However the equivalence results are the same:

CC $\Rightarrow$ NC, CC $\Rightarrow$ R, and NC & R $\Rightarrow$ CC.

Unfortunately, due to the possibility of non-invertible state transition matrices, the equivalence results for LTI systems with unbounded controls are not as strong as for the continuous-time systems.

2.1 LTI Systems with Unbounded Controls

For the system

$$\mathbf{x}_{k+1} = A\,\mathbf{x}_k + B\,\mathbf{u}_k \tag{12}$$

it is only possible to conclude that R $\Rightarrow$ CC. The system of EQ(11) has solution

$$\mathbf{x}_k = A^k\mathbf{x}_0 + \sum_{i=0}^{k-1} A^{k-1-i}B\mathbf{u}_i \tag{13}$$

If A is nilpotent (i.e. there exists m such $A^m = 0$) and so $\mathbf{x}_m = \mathbf{0}$ with no control i.e. with $\mathbf{u}_i = \mathbf{0}$, $i = 0, 1, \ldots, m-1$. Thus for such a system the

[2] One very useful property of $\hat{A}$ is that its characteristic polynomial is $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ which can be obtained by inspection.

ability to be “driven” to $\mathbf{0}$ (null controllability) does not imply the ability to drive the system between arbitrary states.
On the other hand, if the system is R, then for any $\tilde{\mathbf{x}}_f$ there exists a control sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{N-1}$ which satisfies

$$\tilde{\mathbf{x}}_f = \sum_{i=0}^{N-1} A^{N-1-i} B \mathbf{u}_i \qquad (14)$$

In particular, if $\tilde{\mathbf{x}}_f = \mathbf{x}_f - A^N \mathbf{x}_0$, then this same control sequence drives (arbitrary) $\mathbf{x}_0$ to (arbitrary) $\mathbf{x}_f$.

2.2 LTI Systems with unbounded controls: Controllability Criteria and Canonical Forms

The Modal and Rank criteria and the PBH test carry over directly from the continuous-time case along with the accompanying concepts of controllable modes and states and CCF. The only additional point of interest is in the derivation of the Rank criterion in the discrete-time case and its implications for construction of a control sequence.

A consequence of the *Cayley-Hamilton* theorem is that

$$A^k = \sum_{j=0}^{n-1} c_j(k) A^j$$

where the $c_j(k)$ are real scalars. Substituting into EQ(13) gives

$$\begin{aligned} \tilde{\mathbf{x}}_f &= \sum_{i=0}^{N-1} \left(\sum_{j=0}^{n-1} c_j(N-1-i) A^j \right) B \mathbf{u}_i \\ &= \sum_{j=0}^{n-1} A^j B \left(\sum_{i=0}^{N-1} c_j(N-1-i) \mathbf{u}_i \right) \\ &= \left[B, AB, A^2B, \ldots, A^{n-1}B \right] \tilde{\mathbf{u}} \\ &= U \tilde{\mathbf{u}} \end{aligned} \qquad (15)$$

where U is the controllability matrix (as before), $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_{n-1})^T$ and $\tilde{\mathbf{u}}_j = \sum_{i=0}^{N-1} c_j(N-1-i)\mathbf{u}_i$.

Thus the criterion follows. Note that the unknowns ($\tilde{\mathbf{u}}$) in EQ(15) are control values unlike the unknowns in the continuous-time equivalent EQ(6). They have the interpretation that $\tilde{\mathbf{u}}_j$ is the control at time j. The other implication of EQ(15) is that if $\mathbf{x}_f$ is reachable, it is reachable in at most n steps, where n is the system order.