Switched Systems

Given a family of functions $\mathbf{f}_j : \mathbb{R}^n \to \mathbb{R}^n$ labelled by $j \in J$, this in turn gives rise to a family of systems

$$\dot{\mathbf{x}} = \mathbf{f}_j(\mathbf{x}), \qquad j \in J.$$

If we restrict our attention to the case where all the functions are LTI and the set J is finite, this becomes

$$\dot{\mathbf{x}} = A_j\mathbf{x} \qquad j \in J \triangleq \{1, 2, \ldots, p\}$$

where A_j is a $n \times n$ matrix of real entries. To define a switched system generated by the above family, we can consider a switching signal which is a piecewise constant function $\sigma : \mathbb{R} \to J$, the role of which is to specify which subsystem [labelled by $\sigma(t)$] is active at time t. Thus a switched linear system with time-dependent switching is denoted by

$$\dot{\mathbf{x}} = A_{\sigma(t)}\mathbf{x}, \qquad \sigma(t) \in \{1, 2, \ldots, p\}.$$

We'll simplify the notation and write

$$\dot{\mathbf{x}} = A_\sigma\mathbf{x}, \qquad \sigma \in \{1, 2, \ldots, p\} \tag{1}$$

In addition to time-dependent switching, we can also consider state-dependent switching, i.e. a piecewise constant function $\tilde{\sigma} : \mathbb{R}^n \to J$, the role of which is to specify which subsystem [labelled by $\tilde{\sigma}(\mathbf{x})$] is active when the system is in state $\mathbf{x}$. By an abuse of notation, we will refer to both types of switched system by Eq (1).
In passing, we note that switched systems with controlled switching can be described in standard control notation by

$$\dot{\mathbf{x}} = \sum_{j=1}^{p} A_j\mathbf{x}u_j$$

where $u_k = 1$ if $\sigma = k$ and $u_j = 0$ if $j \neq k$.

There are two main problems:
Problem 1 *Find conditions which guarantee the asymptotic stability of the switched system of Eq (1) for arbitrary switching signals.*
It is interesting to note that even when all the subsystems of Eq (1) are asymptotically stable, it is in general not possible to guarantee that the switched system is asymptotically stable for arbitrary switching signals.

Example 1 Consider the switched LTI system with 2 subsystems defined by

$$A_1 = \begin{pmatrix} -0.1 & -1 \\ 2 & -0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -0.1 & -2 \\ 1 & -0.1 \end{pmatrix}$$

Viewed as linear systems, both of these are stable foci (spirals) with eigenvalues $\lambda = -0.1 \pm i\sqrt{2}$. Typical trajectories for each subsystem are shown individually in Fig. 1 and superimposed in Fig. 2.

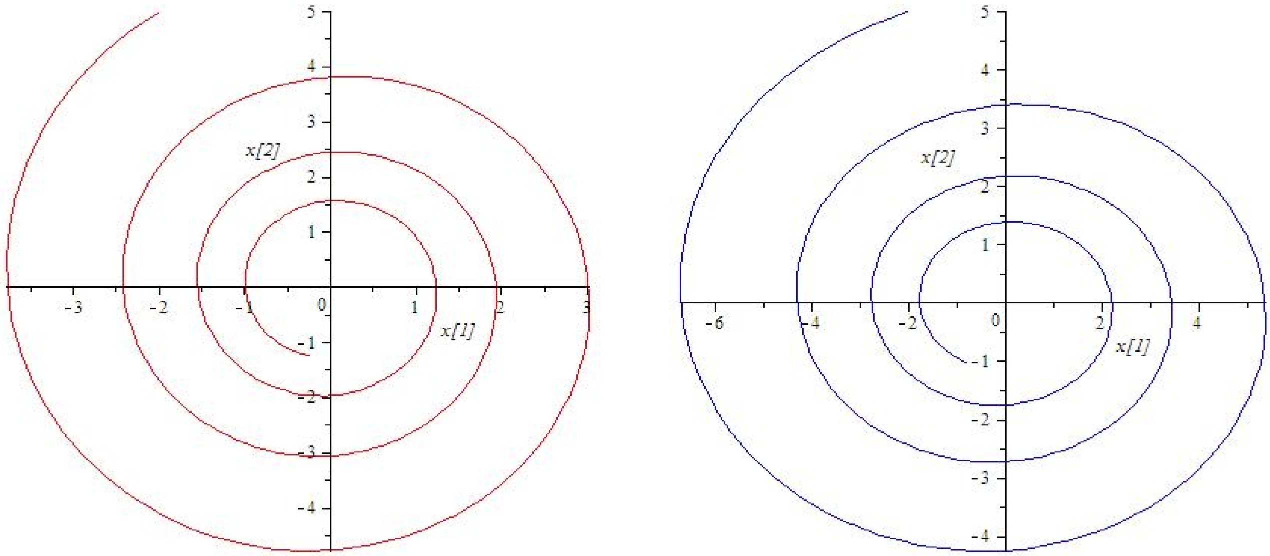

Figure 1: Both trajectories start at $(-2, 5)$: subsystem A_1 (red) and subsystem A_2 (blue).

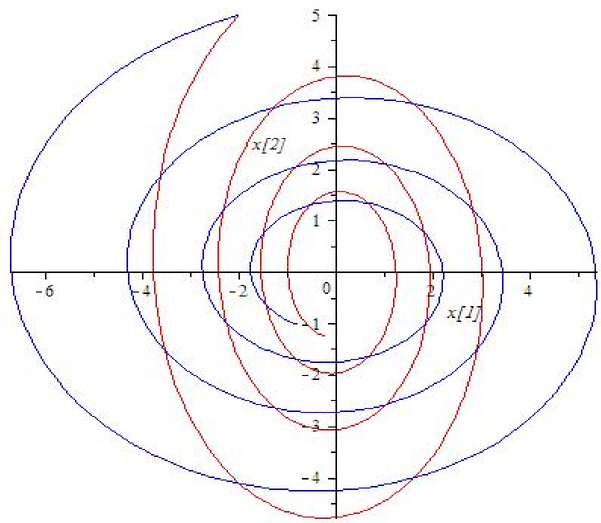

Figure 2: The trajectories of Fig. 1 superimposed on each other.

We can define a switched system by the state-dependent switching rule "use subsystem A_1 if $x_1 x_2 \leq 0$, otherwise use subsystem A_2". We'll call this *Switched System 1-1.* We can define a second switched system by "use subsystem A_1 if $x_1 x_2 > 0$, otherwise use subsystem A_2". We'll call this *Switched System 1-2.* Typical trajectories for these systems are shown in Fig. 3. The trajectory of Switched System 1-1 is asymptotically stable, while that of Switched System 1-2 is unstable.

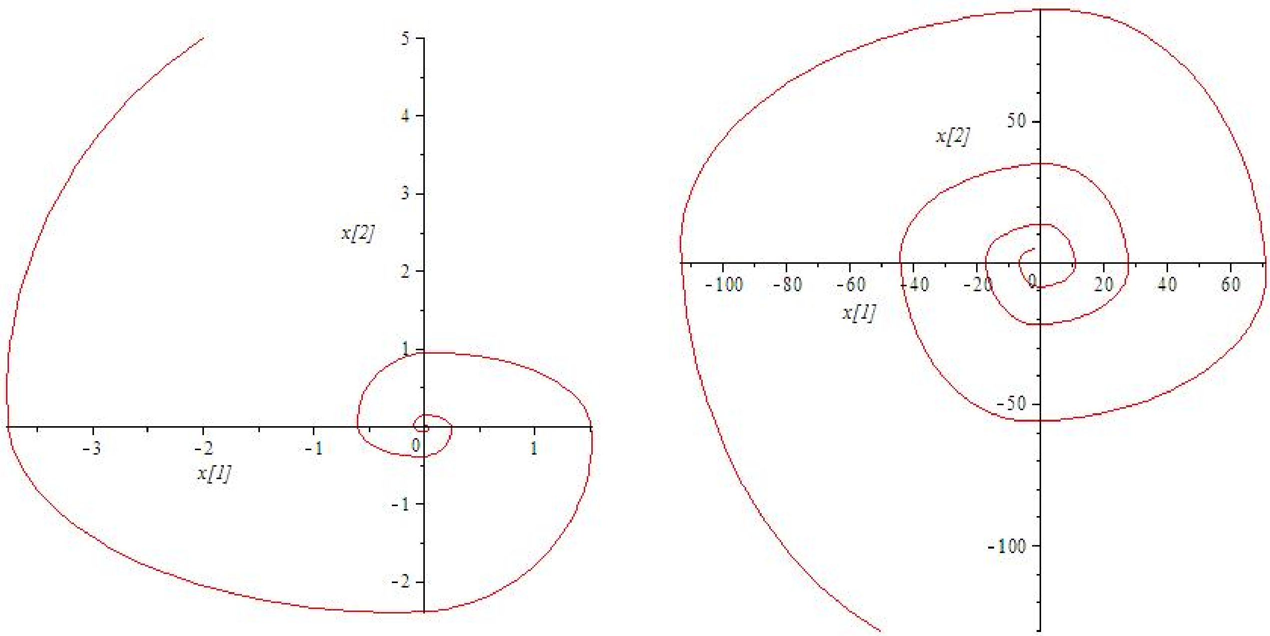

Figure 3: Trajectory of Switched System 1-1 (left), that of Switched System 1-2 (right). Both trajectories start at $(-2, 5)$.

Problem 2 *If the switched system of Eq (1) is not asymptotically stable for arbitrary switching signals, identify those switching signals for which it is asymptotically stable.* Again it is interesting to note that even when all the subsystems of Eq (1) are unstable, it is sometimes possible to guarantee asymptotic stability of the switched system.

Example 2 Consider the switched LTI system with 2 subsystems defined by

$$\mathcal{A}_1 = \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0.1 & -2 \\ 1 & 0.1 \end{pmatrix}$$

Viewed as linear systems, both of these are unstable foci with eigenvalues $\lambda = 0.1 \pm i\sqrt{2}$. Typical trajectories for each subsystem are shown individually in Fig. 4 and superimposed in Fig. 5.

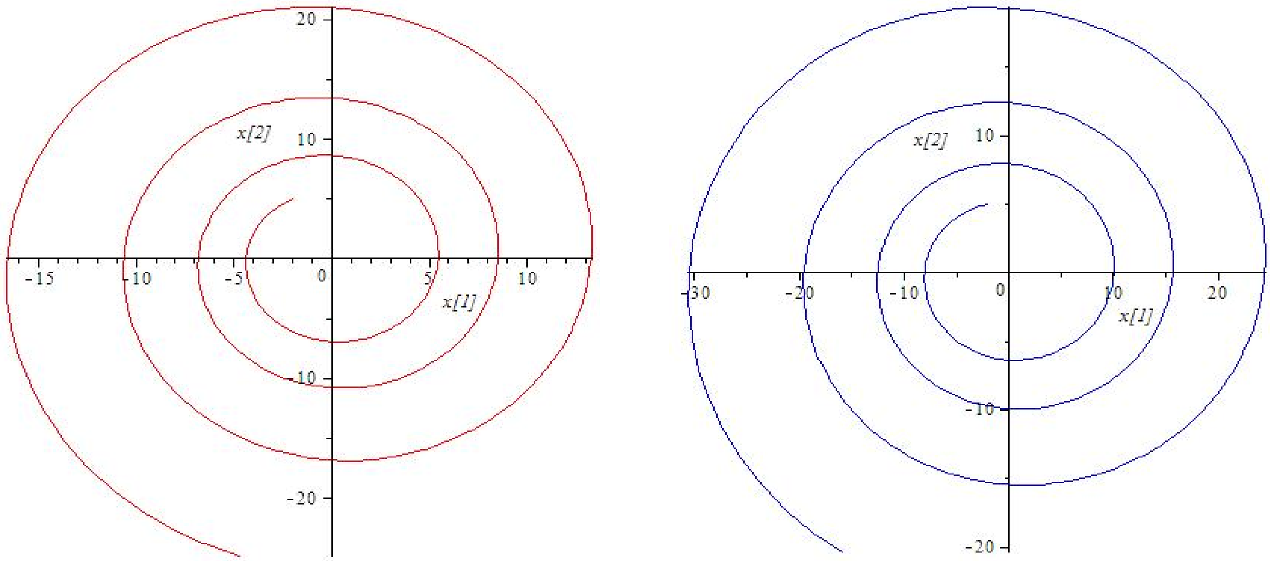

Figure 4: Both trajectories start at $(-2, 5)$: subsystem $\mathcal{A}_1$ (red) and subsystem $\mathcal{A}_2$ (blue).

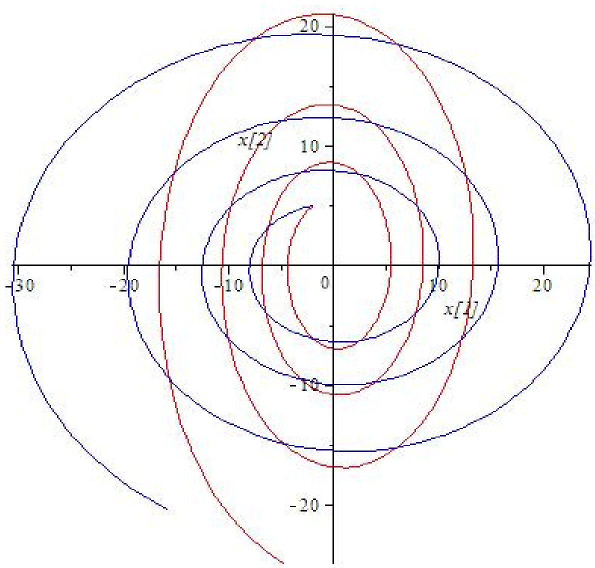

Figure 5: The trajectories of Fig. 4 superimposed on each other.

Define a switched system by the state-dependent switching rule "use subsystem $\mathcal{A}_1$ if $x_1x_2 \leq 0$, otherwise use subsystem $\mathcal{A}_2$". We'll call this *Switched System 2-1.* We can define a second switched system by "use subsystem $\mathcal{A}_1$ if $x_1x_2 > 0$, otherwise use subsystem $\mathcal{A}_2$". We'll call this *Switched System 2-2.* Typical trajectories for these systems are shown in Fig. 6. The trajectory of Switched System 2-1 is asymptotically stable, while that of Switched System 2-2 is unstable.

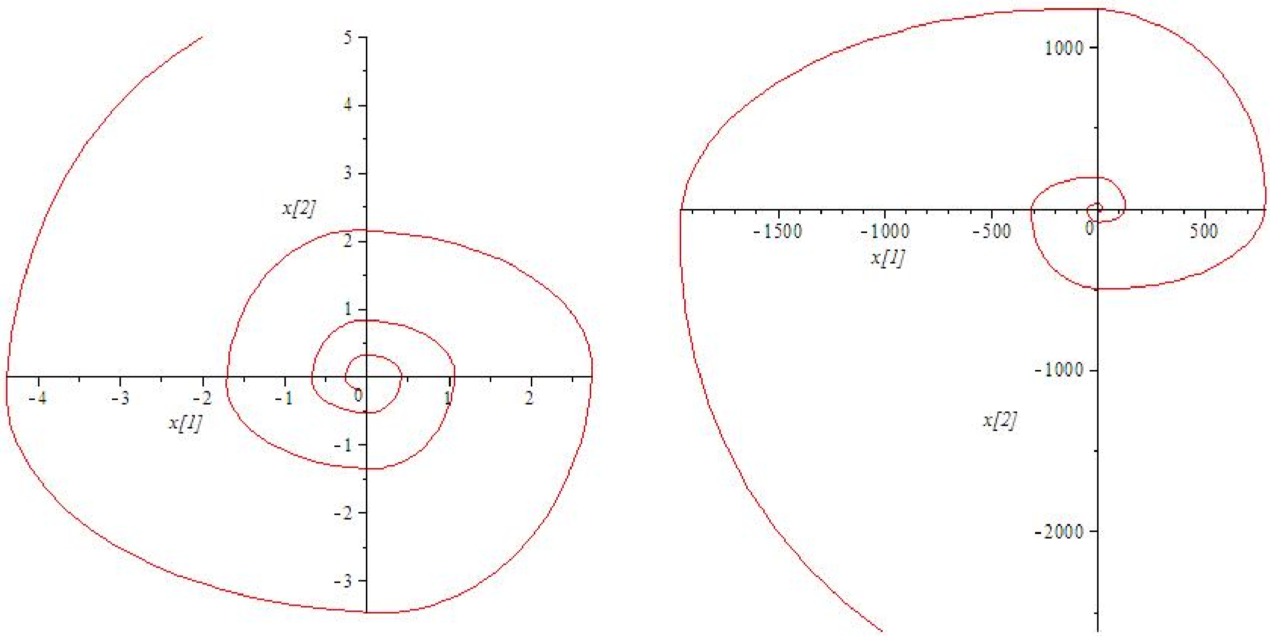

Figure 6: Trajectory of Switched System 2-1 (left), that of Switched System 2-2 (right). Both trajectories start at $(-2, 5)$.

Using the Eigenstructure of Subsystems to investigate the Stability of Switched Systems

Xu & Antsaklis[1] have given necessary and sufficient conditions for the stabilisation of sets of unstable switched 2-d LTI systems. Here we illustrate what can be achieved by a consideration of the eigenstructures of a switched LTI system which is comprised of 2 saddles.

For convenience, we recall that the eigenstructure of a (single) saddle LTI system with some typical trajectories is as shown in Fig. 7.

[1] *Xu Xuping* & P J Antsaklis (1999) Stabilization of Second-Order LTI switched Systems, Technical Report ISIS-99-001, University of Notre Dame.

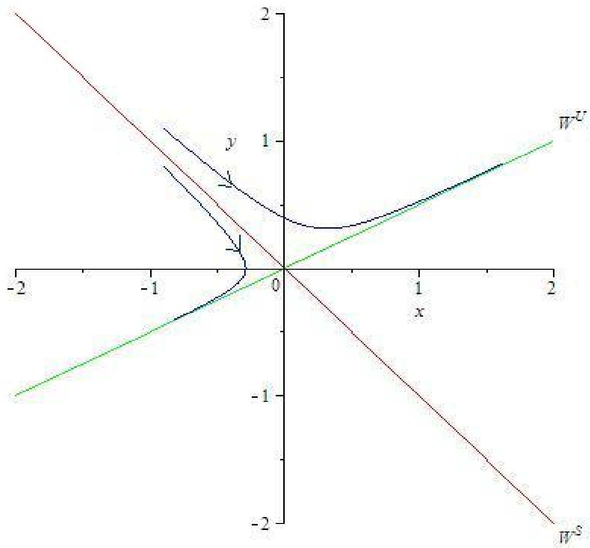

Figure 7: Some trajectories (blue) of a saddle system with stable (red) W^S and unstable (green) W^U manifolds

Fig. 8 shows the invariant manifolds of the switched saddles. In Fig. 8 (Left) the interleaving of the red followed by green followed by red etc. lines is a necessary and sufficient condition for stabilisation to occur. Consider any sector (we'll refer to the subsystems as 1 and 2, and to the boundary lines of a sector as "rays"; notice that the ordering of the rays is 1,1, 2,2, 1,1, 2,2). Sufficiency follows from:

(i) If both rays of the sector are associated with the subsystem 1 - switch on subsystem 2: then the trajectory meets the red boundary of the 1 sector as it's being attracted to the unstable green manifold of 2 ; switch to 1 to head to the origin.

(ii) If the sector is bounded by a green ray of 1 and a red ray of 2 - switch on 2: then the trajectory crosses the green ray of 1 and meets the red ray of 1 as it's being attracted to the unstable green manifold of 2; at this second intersection switch to 1 to head to the origin.

On the other hand, if the red-green rays don't interleave as in Fig. 8 (Left), then there must be a sector bounded by two green rays (See Fig. 8 (Right) for an illustration): any trajectory originating in this sector cannot escape this sector using either 1 or 2 as the trajectories must be asymptotic to one or other of the green rays and so flee the origin.

The stabilising switching described above is"non robust", as it is virtually impossible to switch exactly on a manifold (See Fig. 9 for an example). A much more realistic strategy is to switch just "beyond" the red curves. An alternative is to use the "conical" switching strategies given in *Xu* & Antsaklis.

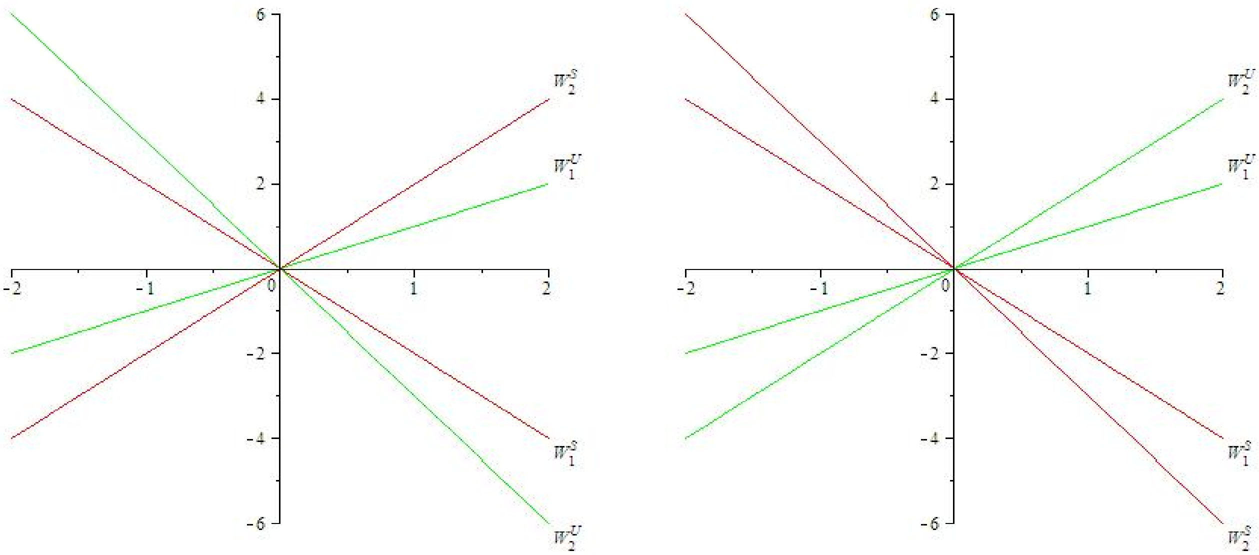

Figure 8: W_i^S and W_i^U are the stable (red) and unstable (green) manifolds respectively of subsystem i.

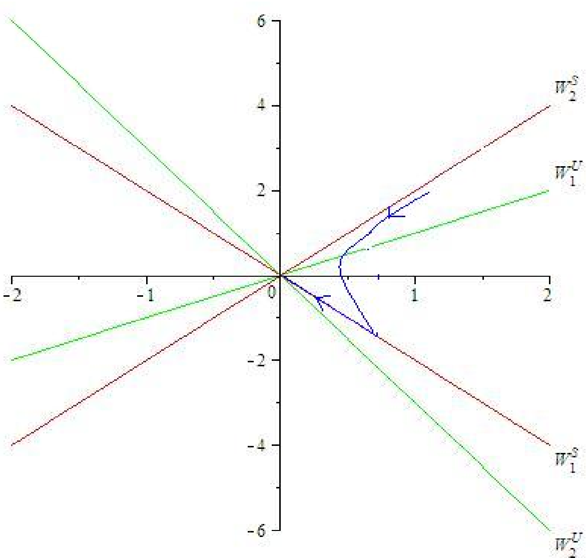

Figure 9: An example of a trajectory associated with a non robust switching strategy