Neimark-Sacker Bifurcation

The *Neimark-Sacker* bifurcation (NSB) is the "equivalent" of the *Hopf* bifurcation for maps. For instance, in the case of a supercritical NSB, a stable focus loses its stability as a parameter is varied with the consequent birth of a stable cycle or quasi-cycle - we'll refer to either of these as a closed invariant curve. In the case of a subcritical NSB, a stable focus enclosed by an unstable closed curve loses its stability with the consequent disappearance of the closed invariant curve as a parameter is varied.

The criterion for checking whether the bifurcation occurs is similar in spirit to that of the *Hopf* case with a few minor differences. This note concentrates on the 2-d case.

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, a), \qquad \mathbf{x}_e = \mathbf{f}(\mathbf{x}_e, a)$$

Assume $\mathbf{x}_e(a)$ exists over an interval of a-values of interest, and furthermore the *Jacobian* matrix of the map at $\mathbf{x} = \mathbf{x}_e$ has the pair of eigenvalues $\lambda(a) = R(a)e^{i\theta(a)}$.

If at $a = a_c$,

$$R(a_c) = 1, \qquad 0 < \theta(a_c) < \pi \tag{1}$$

and the non-degeneracy conditions hold

$$\frac{dR}{da}(a_c) \neq 0, \qquad e^{ik\theta(a_c)} \neq 1, k = 1, 2, 3, 4 \tag{2}$$

then a NSB occurs at $a = a_c$.

Consider the map

$$x' = ax(1 - y), \qquad y' = x$$

It has fixed points $\mathbf{x}_e = \mathbf{0}$ and $\mathbf{x}_e = \overline{\mathbf{x}} = (1 - 1/a, 1 - 1/a)^T$.
At $\mathbf{x}_e = \overline{\mathbf{x}}$, $\lambda(a) = (\sqrt{a-1})e^{i\tan^{-1}(\sqrt{4a-5})}$; thus we have

$$R(a) = \sqrt{a-1}, \qquad \theta(a) = \tan^{-1}(\sqrt{4a-5})$$

Setting $R(a_c) = 1$ gives $a_c = 2$; and so $\theta(a_c) = \pi/3$.
Checking the non-degeneracy conditions, we get

$$\frac{dR}{da}(a_c) = \frac{1}{2\sqrt{2}} \neq 0$$

$$e^{ik\theta(a_c)} = e^{ik\frac{\pi}{3}} = \frac{1 + i\sqrt{3}}{2}, \quad \frac{-1 + i\sqrt{3}}{2}, \quad -1, \quad \frac{-1 - i\sqrt{3}}{2} \quad \neq 1$$

for $k = 1, 2, 3, 4$ respectively.

Is the bifurcation super- or subcritical? See the accompanying *Maple* worksheet "*Neimark-Sacker* Bifurcation", where a simulation seems to indicate that it is supercritical.