Hopf Bifurcation

A *Hopf* bifurcation [1] is a local bifurcation in which a focus (fixed point) of a continuous-time system (flow) changes stability as a parameter is varied. Under reasonably generic assumptions about the structure of the flow, a limit cycle branching from the fixed point is born. The bifurcation can be supercritical or subcritical resulting in stable or unstable limit cycles respectively. (See Figs. 1 and 2 respectively).

Andronov-Hopf **Theorem**

Consider the flow

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p), \qquad \mathbf{x} \in \mathbb{R}^n, \;\; p \in \mathbb{R}$$

where $\mathbf{f}$ is smooth.
If (i) there exists a fixed point $\mathbf{x}_e(p)$ for $p \in I(\text{interval}) \subset \mathbb{R}$ and (ii) at $\mathbf{x}_e(p)$, the system has Jacobian $J(\mathbf{x}_e)$ with one pair of complex conjugate eigenvalues $\lambda(p) = \Re(p) \pm i\Im(p)$ for which, at a critical value of $p \; (= p_c) \in I$,

- $\Re(p_c) = 0$
- $\Im(p_c) \neq 0$
- $\frac{d\Re}{dp}(p_c) \neq 0$

then, as p passes through p_c, $\mathbf{x}_e(p)$ changes stability and a limit cycle bifurcates from it. Furthermore the period of the limit cycle approaches $2\pi/\Im(p_c)$ as $p \to p_c$.

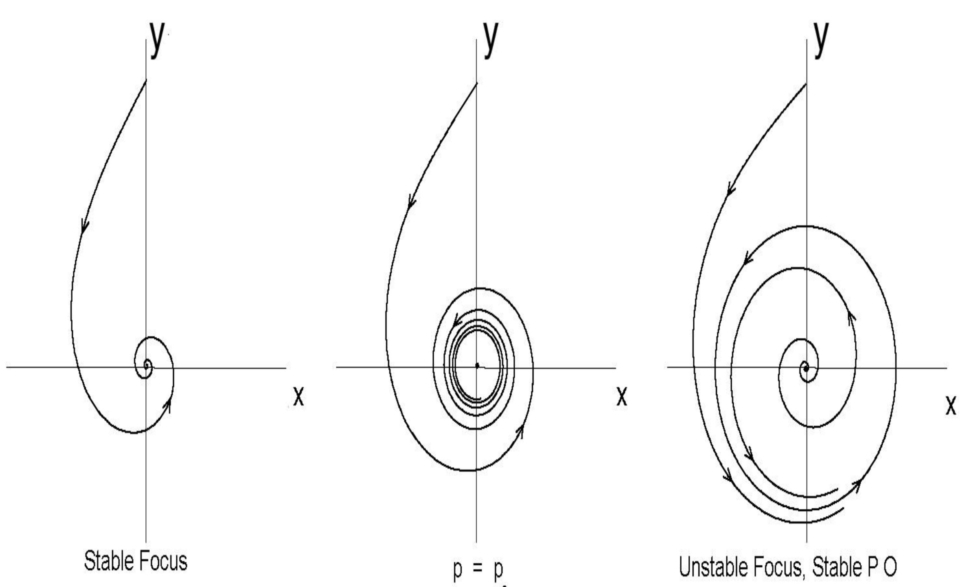

Figure 1: Supercritical Hopf bifurcations. (P O = periodic orbit)

[1]It is sometimes referred to as a *Andronov-Hopf* or a *Poincaré-Andronov-Hopf* bifurcation

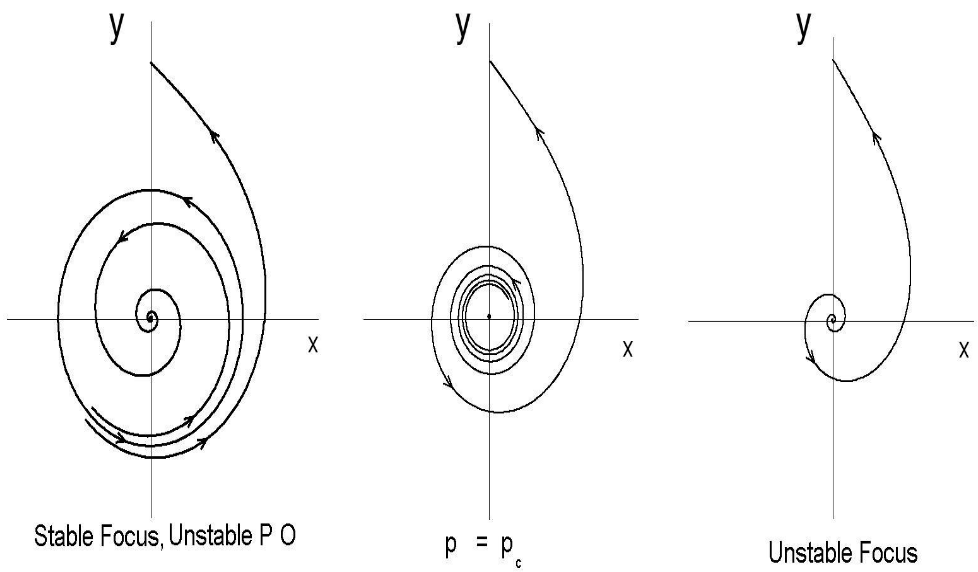

Figure 2: Subcritical Hopf bifurcations. (P O = periodic orbit)

Example: *Van der Pol* Oscillator
The flow

$$\dot{x} = \mu(1 - y^2)x - y, \qquad \dot{y} = x \tag{1}$$

has a fixed point at the origin ($\mathbf{x}_e = \mathbf{0}$) for all values of the parameter μ. The Jacobian matrix evaluated at the origin is

$$J = \begin{bmatrix} \mu & -1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

with eigenvalues

$$\begin{aligned} \lambda &= \Re(\mu) \pm i\Im(\mu) \\ &= \frac{\mu}{2} \pm i\sqrt{1 - \left(\frac{\mu}{2}\right)^2} \end{aligned} \tag{3}$$

for $|\mu| < 2$.
At $\mu = \mu_c \triangleq 0$ we have

- $\Re(\mu_c) = 0$
- $\Im(\mu_c) = 1 \neq 0$
- $\frac{d\Re}{d\mu}(\mu_c) = \frac{1}{2} \neq 0$

So, a *Hopf* bifurcation takes place at $\mu = \mu_c$. Notice that $\mathbf{x}_e$ is (i) stable for $\mu < \mu_c$ and (ii) unstable for $\mu > \mu_c$. It is known that this bifurcation is supercritical.