

## Cournot Duopoly with Homogeneous items: Linear Demand and Linear Costs

Let  $x_1$  and  $x_2$  be the quantities of homogeneous items produced by two firms with associated costs  $C_1(x_1) = c_1x_1$  and  $C_2(x_2) = c_2x_2$  respectively.

Items sell at  $P = a - b(x_1 + x_2)$  each and it is assumed that all items produced are sold. The profits made by the firms are then

$$\pi_1 = Px_1 - c_1x_1 = (a - c_1 - b(x_1 + x_2))x_1$$

$$\pi_2 = Px_2 - c_2x_2 = (a - c_2 - b(x_1 + x_2))x_2$$

respectively.

Let's consider the specific example:  $P = 5 - \frac{x_1+x_2}{2000}$  where the firms have costs  $C_1(x_1) = 2x_1$  and  $C_2(x_2) = 2.5x_2$  respectively.

The profits made by the firms are then

$$\pi_1 = \left(3 - \frac{x_1 + x_2}{2000}\right)x_1$$

$$\pi_2 = \left(2.5 - \frac{x_1 + x_2}{2000}\right)x_2$$

respectively.

Maximising  $\pi_1$  with respect to  $x_1$

$$\begin{aligned} \frac{\partial \pi_1}{\partial x_1} &= 3 - \frac{x_1 + x_2}{2000} - \frac{1}{2000}x_1 \\ &\stackrel{set}{=} 0 \\ \Rightarrow x_1 &= 3000 - \frac{1}{2}x_2 \end{aligned} \tag{1}$$

Similarly maximising  $\pi_2$  with respect to  $x_2$  yields

$$x_2 = 2500 - \frac{1}{2}x_1 \tag{2}$$

Equations 1 and 2 are referred to as *Reaction Functions* or Best Response Functions - provided their solutions are nonnegative, which I'll assume in the following.

Solving equations 1 and 2 simultaneously gives the *equilibrium* values

$$x_1^* = 2333.33, \quad x_2^* = 1333.33$$

At these equilibrium values

$$P^* = 3.17$$

and

$$\pi_1^* = 2722.22, \quad \pi_2^* = 888.89 \tag{3}$$

*Cournot* duopoly is an example of a 2-player matrix form game with an infinite number of strategies available to both players (firms), i.e. the choice of  $x_1$  and  $x_2$  respectively.  $\langle x_1^*, x_2^* \rangle$  is then a *Nash* equilibrium with payoffs  $\pi_1^*$  and  $\pi_2^*$  respectively.

## Stackelberg Duopoly

*Stackelberg* duopoly is an example of a 2-player extensive form game in which Firm 1 moves first (the "Leader") and Firm 2 responds (the "Follower"). Irrespective of what the leader does, the follower will use the reaction function (Eq. 2) as it is its best response.

Knowing this, the leader seeks to maximise

$$\Pi_1 = \left( 3 - \frac{(x_1 + 2500 - \frac{1}{2}x_1)}{2000} \right) x_1 = \left( 1.75 - \frac{x_1}{4000} \right) x_1$$

as a function of  $x_1$ .

$$\begin{aligned} \frac{d\Pi_1}{dx_1} &= 1.75 - \frac{x_1}{2000} \\ &\stackrel{set}{=} 0 \\ \Rightarrow x_1 &= 3500 \end{aligned} \tag{4}$$

Denoting this optimal value by  $X_1^*$  and the corresponding value of  $x_2$  by  $X_2^*$  (substitute Eq. 4 into Eq. 2) gives

$$X_1^* = 3500, \quad X_2^* = 750$$

At these equilibrium values

$$P^* = 2.875$$

and

$$\Pi_1^* = 3062.5, \quad \Pi_2^* = 281.25 \tag{5}$$

Exercise: Redo the analysis if Firm 2 is the leader?