Cournot Duopoly with Heterogeneous items: Linear Demand and Linear Costs

Let x_1 and x_2 be the quantities of heterogeneous or non-identical items produced by two firms with associated costs $C_1(x_1) = c_1x_1$ and $C_2(x_2) = c_2x_2$ respectively.
Firm 1's items sell at $P_1 = a_1 - b_{11}x_1 - b_{12}x_2$ each, firm 2's at $P_2 = a_2 - b_{21}x_1 - b_{22}x_2$ each and it is assumed that all items produced are sold.
The profits made by the firms are then

$$\pi_1 = P_1x_1 - c_1x_1 = (a_1 - c_1 - b_{11}x_1 - b_{12}x_2)\, x_1$$

$$\pi_2 = P_2x_2 - c_2x_2 = (a_2 - c_2 - b_{21}x_1 - b_{22}x_2)\, x_2$$

respectively.
We'll proceed with a specific example: $P_1 = 10 - \frac{x_1+x_2}{1000}$, $P_2 = 9 - \frac{x_1+x_2}{1250}$, $C_1(x_1) = 2.5x_1$, $C_2(x_2) = 2x_2$. The profits made by the firms are then

$$\pi_1 = P_1x_1 - c_1x_1 = \left(7.5 - \frac{x_1 + x_2}{1000}\right) x_1$$

$$\pi_2 = P_2x_2 - c_2x_2 = \left(7 - \frac{x_1 + x_2}{1250}\right) x_2$$

respectively.
Maximising π_1 with respect to x_1

$$\begin{aligned} \frac{\partial \pi_1}{\partial x_1} &= 7.5 - \frac{2}{1000}x_1 - \frac{1}{1000}x_2 \\ &\stackrel{set}{=} 0 \\ \Rightarrow x_1 &= 3750 - \frac{1}{2}x_2 \end{aligned} \tag{1}$$

Similarly maximising π_2 with respect to x_2 yields

$$x_2 = 4375 - \frac{1}{2}x_1 \tag{2}$$

We'll assume that Equations 1 and 2 give nonnegative values for x_1 and x_2 and so represent Reaction Functions. Solving equations 1 and 2 simultaneously gives the *equilibrium* values

$$\begin{aligned} x_1^* &= 2083.33 \\ x_2^* &= 3333.33 \end{aligned}$$

At these equilibrium values

$$P_1^* = 4.5833, \qquad P_2^* = 4.67$$

$$\pi_1^* = 4340.28, \qquad \pi_2^* = 8888.89$$

Stackelberg Duopoly

We'll consider a *Stackelberg* duopoly in which Firm 1 is the Leader and Firm 2 is the Follower. Irrespective of what the leader does, the follower will use the reaction function (Eq. 2) as its best response.

Knowing this, the leader seeks to maximise

$$\Pi_1 = \left(7.5 - \frac{x_1 + 4375 - \frac{1}{2}x_1}{1000}\right) x_1 = \left(3.125 - \frac{x_1}{2000}\right) x_1$$

as a function of x_1.

$$\begin{aligned} \frac{d\Pi_1}{dx_1} &= 3.125 - \frac{x_1}{1000} \\ &\stackrel{set}{=} 0 \\ \Rightarrow x_1 &= 3125 \end{aligned} \tag{3}$$

Denoting this optimal value by X_1^* and the corresponding value of x_2 by X_2^* (substitute Eq. 3 into Eq. 2) gives

$$X_1^* = 3125, \qquad X_2^* = 2812.5$$

At these equilibrium values

$$P_1^* = 4.0625, \qquad P_2^* = 4.25$$

and

$$\Pi_1^* = 4882.8125, \qquad \Pi_2^* = 6328.125 \tag{4}$$