# Graphical Analysis of 1-d Monotone Maps

It is well known that (strictly) increasing maps can only have fixed points, while decreasing maps can have fixed points and period-2 points in addition. In certain cases we can find estimates of the basins of attraction of stable fixed points, and “basins of repulsion” of unstable fixed points. Clark Robinson states and proves a well-known result for increasing functions with a stable fixed point([1], Theorem 9.4.2)

**Theorem 1: Increasing Map with stable fixed point**. Let $f$ be continuous, increasing and have a fixed point $\overline{x}$ in $(z_1,\ z_2)$. Furthermore if

$$x < f(x) < \overline{x} \qquad \text{for } z_1 < x < \overline{x}$$

and

$$\overline{x} < f(x) < x \qquad \text{for } \overline{x} < x < z_2$$

then the basin of attraction of $\overline{x}$ includes $(z_1,\ z_2)$. (See Fig. 1: in each of this and the following figures, the red curve within the blue region represents a function which satisfies the hypotheses of the theorem being illustrated in the diagram).

Proof (after [1], p.352) Consider $x_0 \in (z_1,\ \overline{x})$. By repeated application of $f$, we generate the sequence $x_n = f^n(x_0)$ and with $x < f(x) < \overline{x}$, we get

$$x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n < \overline{x}$$

Since this sequence is increasing and bounded above (by $\overline{x}$), it converges to $x_\infty$, say; but $x' = f(x)$ means that $x_\infty$ is a fixed point and since $\overline{x}$ is the only fixed point in the interval $(z_1,\ \overline{x}]$, $x_\infty = \overline{x}$. Furthermore since $x_0$ is an arbitrary point in $(z_1,\ \overline{x})$, we have that $(z_1,\ \overline{x})$ is contained in the basin of attraction of $\overline{x}$.
Similarly by selecting an initial point in $(\overline{x},\ z_2)$, we can show that this latter interval is contained in the basin of attraction of $\overline{x}$.

In a similar vein, we can establish the following result for an unstable fixed point.
**Theorem 2: Increasing Map with unstable fixed point**. Let $f$ be continuous, increasing and have a fixed point $\overline{x}$ in $(z_1,\ z_2)$. Furthermore if

$$f(x) < x \qquad \text{for } z_1 < x < \overline{x}$$

and

$$x < f(x) \qquad \text{for } \overline{x} < x < z_2$$

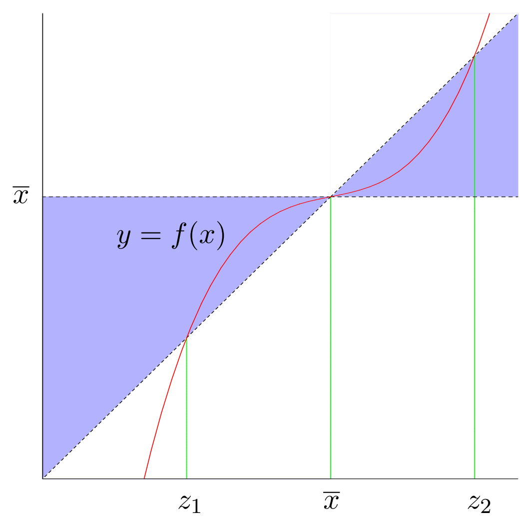


Figure 1: Increasing map with stable fixed point

then the basin of repulsion of $\overline{x}$ includes $(z_1,\ z_2) \setminus \{\overline{x}\}$, i.e. any trajectory originating in this set flees the interval $(z_1,\ z_2)$. (See Fig. 2)

Proof: Consider $x_0 \in (z_1,\ \overline{x})$. By repeated application of $f$, we again generate the sequence $x_n = f^n(x_0)$ and with $f(x) < x$, we get

$$x_n < x_{n-1} < \cdots < x_1 < x_0 < \overline{x}.$$

This decreasing sequence either converges to a fixed point $x_\infty$ which is outside the interval $(z_1,\ \overline{x})$ since $\overline{x}$ is a fixed point which is not in this interval, or diverges (to $-\infty$). In either case, the trajectory flees the interval $(z_1,\ \overline{x})$. In particular, there exists $N$ such that $\forall n > N, x_n \leq z_1$.

Similarly by selecting an initial point in $(\overline{x},\ z_2)$, we can show that this latter interval is contained in the basin of repulsion for $\overline{x}$. Again, in particular, there exists $N$ such that $\forall n > N, x_n \geq z_2$.

Similar results hold for decreasing maps.

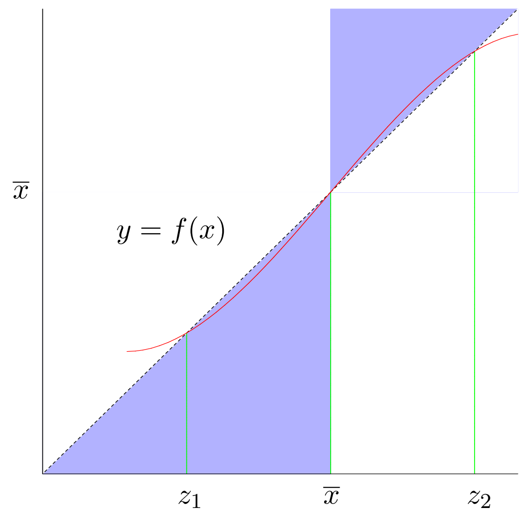


Figure 2: Increasing map with unstable fixed point

**Theorem 3: Decreasing Map with stable fixed point**. Let $f$ be continuous, decreasing and have a fixed point $\overline{x}$ in $(z_1,\ z_2)$. Furthermore if

$$\overline{x} < f(x) < -x + 2\overline{x} \qquad \text{for } z_1 < x < \overline{x}$$

and

$$-x + 2\overline{x} < f(x) < \overline{x} \qquad \text{for } \overline{x} < x < z_2$$

then the basin of attraction of $\overline{x}$ includes $(z_1,\ z_2)$. (See Fig. 3)
Proof: We note that the map $f^2 = f \circ f$ is increasing. Furthermore

$$\begin{aligned} z_1 &< x < \overline{x} \\ \Rightarrow \overline{x} &< f(x) < -x + 2\overline{x} \\ \Rightarrow \overline{x} = f(\overline{x}) &> f^2(x) > f(-x + 2\overline{x}) > -x + 2\overline{x} > z_1 \end{aligned}$$

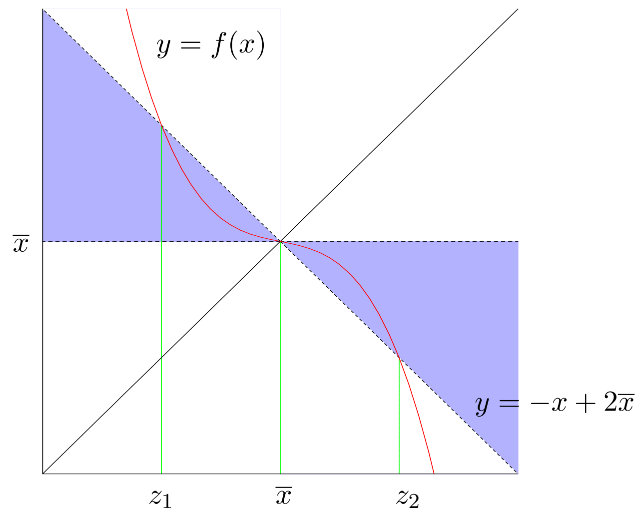


Figure 3: Decreasing map with stable fixed point

and similarly

$$\begin{aligned} \overline{x} &< x < z_2 \\ \Rightarrow -x + 2\overline{x} &< f(x) < \overline{x} \\ \Rightarrow z_2 > -x + 2\overline{x} > f(-x + 2\overline{x}) &> f^2(x) > f(\overline{x}) = \overline{x} \end{aligned}$$

Thus $f^2$ satisfies the hypotheses of Theorem 1 and so we can apply the theorem to get the conclusion we want.

**Theorem 4: Decreasing Map with unstable fixed point**. Let $f$ be continuous, decreasing and have a fixed point $\overline{x}$ in $(z_1,\ z_2)$. Furthermore if

$$-x + 2\overline{x} < f(x) \qquad \text{for } z_1 < x < \overline{x}$$

and

$$f(x) < -x + 2\overline{x} \qquad \text{for } \overline{x} < x < z_2$$

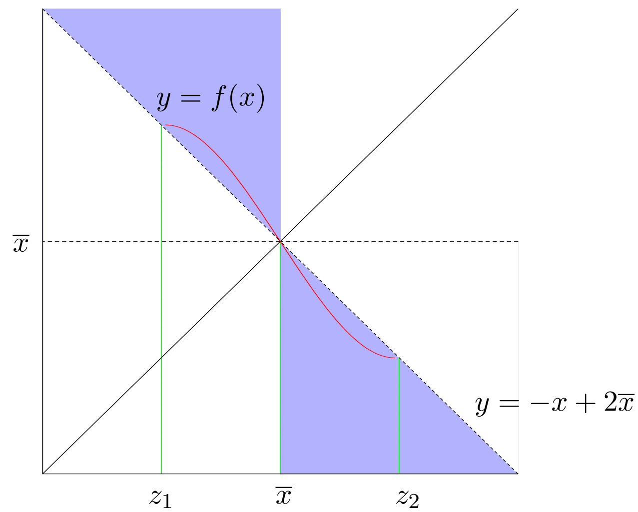


Figure 4: Decreasing map with unstable fixed point

then the basin of repulsion of $\overline{x}$ includes $(z_1,\ z_2) \setminus \{\overline{x}\}$. (See Fig. 4)
Proof: as per the previous theorem, it can be shown that $f^2$ is increasing, and satisfies the hypotheses of Theorem 2, and so the conclusion follows.

# References

[1] R. Clark Robinson, *An Introduction to Dynamical Systems*, Prentice Hall, New Jersey, (2004)