Flows & Phase Space

1 Flows

We are assuming that the IVP

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

has a unique solution [1]

$$\mathbf{x}(t) = \Phi(t, \mathbf{x}_0), \qquad -\tau < t < \tau. \tag{2}$$

This solution is also called a trajectory, orbit or flow. By an abuse of terminology, we also call Eqn (1) a flow.
From Eqn (1) we see that the vector tangent to the flow, the " tangent vector", is $\dot{\mathbf{x}}$ or $\mathbf{f}$ (See Fig 1).

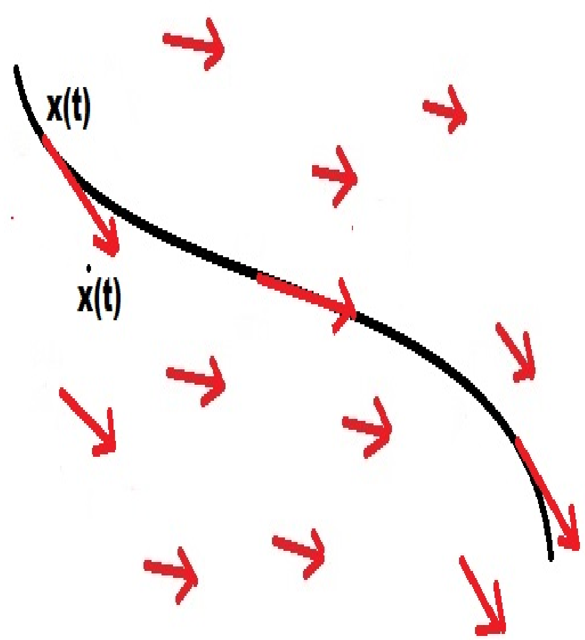

Figure 1: Flow and Tangent Vectors in Phase Space

2 Uniqueness

Uniqueness of solutions means that trajectories don't intersect in phase space - at a finite time. If they did intersect, at the state $\mathbf{x} = \mathbf{a}$ (say), then where does the trajectory that starts in state $\mathbf{a}$ go? This plays a major role in 2-d systems, as its constrains where trajectories can lie.

[1] Sufficient conditions for this are, for instance, $\mathbf{f}$ is continuously differentiable or $\mathbf{f}$ satisfies a *Lipshitz* condition.

3 Nullclines

The i-th nullcline of Eqn (1) is the hypersurface obtained by solving

$$0 = f_i(\mathbf{x}) \tag{3}$$

In 2-d nullclines are generically curves, and are useful for visualising how trajectories flow in the phase plane. In particular, tangent vectors run vertically along the x- nullcline ($\dot{x} = 0$) and horizontally along the y- nullcline ($\dot{y} = 0$).
Fixed points are situated at the common intersection points of all the nullclines.

4 Limit Sets of $\mathbf{x}_0$

If $\mathbf{x}_0$ generates the trajectory given by Eqn (2), then

- the omega-limit set $\omega(\mathbf{x}_0)$ is the set of points z for which there exist a sequence of increasing times t_j such that $\mathbf{x}(t_j) \to z$ as $t_j \to \infty$.
- the alpha-limit set $\alpha(\mathbf{x}_0)$ is the set of points w for which there exist a sequence of decreasing times t_j such that $\mathbf{x}(t_j) \to w$ as $t_j \to -\infty$.

5 Invariant Manifolds of $\mathbf{x}_e$

Let $\mathbf{x}_e$ be an equilibrium state/fixed point of the flow Eqn (1) i.e.

$$\mathbf{f}(\mathbf{x}_e) = \mathbf{0} \tag{4}$$

If $\mathbf{x}_e$ is hyperbolic [2] , the stable manifold of $\mathbf{x}_e$ is

$$W^S(\mathbf{x}_e) = \{\mathbf{x}_0 \mid \Phi(t, \mathbf{x}_0) \to \mathbf{x}_e \text{ as } t \to \infty\} \tag{5}$$

and the unstable manifold is

$$W^U(\mathbf{x}_e) = \{\mathbf{x}_0 \mid \Phi(t, \mathbf{x}_0) \to \mathbf{x}_e \text{ as } t \to -\infty\} \tag{6}$$

If $\mathbf{x}_e$ is non-hyperbolic, the flow has stable, unstable and centre manifolds. The centre manifold $W^C(\mathbf{x}_e)$ consists of trajectories whose behaviour near $\mathbf{x}_e$ is not controlled by the attraction of $W^S(\mathbf{x}_e)$ nor the repulsion of $W^U(\mathbf{x}_e)$.

[2] The *Jacobian* matrix $D\mathbf{f}(\mathbf{x}_e)$ has no eigenvalue with zero real part.

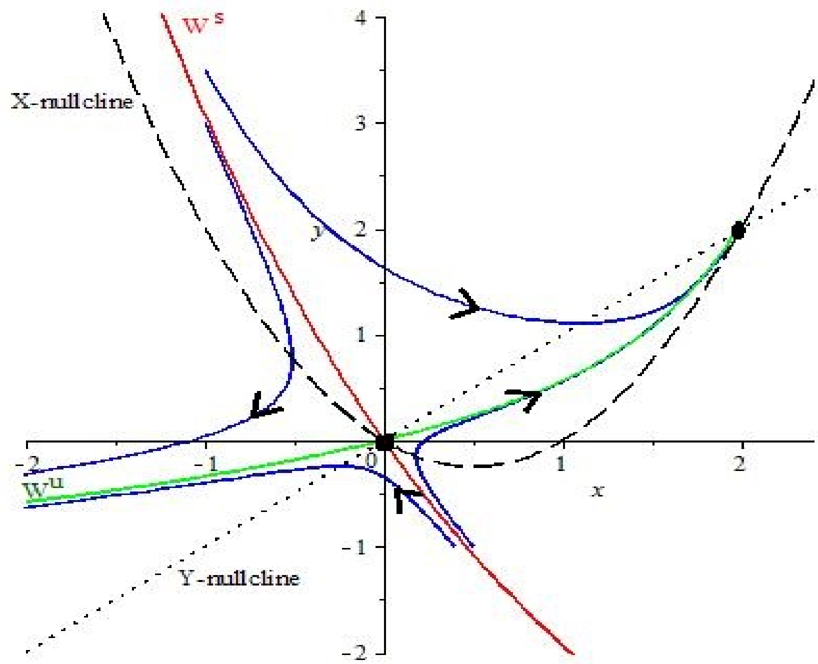

Figure 2: Trajectories, Nullclines and Invariant manifolds for a 2-d flow.

Fig 2 shows phase plane features for the 2-d system

$$\dot{x} = x + y - x^2, \qquad \dot{y} = x - y. \tag{7}$$

This flow has a saddle at $(0,0)^T$ and a stable node at $(2,2)^T$. The stable and unstable manifolds of the origin are shown in Fig 2, while that portion of $W^U(\mathbf{x}_e)$ linking $(0,0)^T$ and $(2,2)^T$ forms a heteroclinic orbit (see Section 6).

Exercises: (1) Draw the tangent vectors along the nullclines. (2) What are the omega- and alpha-limit sets of (i) $(1,0)^T$, (ii) $(-1, 3.0691)^T$ and (iii) $(-1, 0.3397)^T$, given that the latter two points are on the appropriate invariant manifold?

6 Homoclinicity & Heteroclinicity

- A homoclinic orbit is a trajectory which joins a saddle fixed point to itself.
- A heteroclinic orbit is a trajectory which joins two different fixed points.

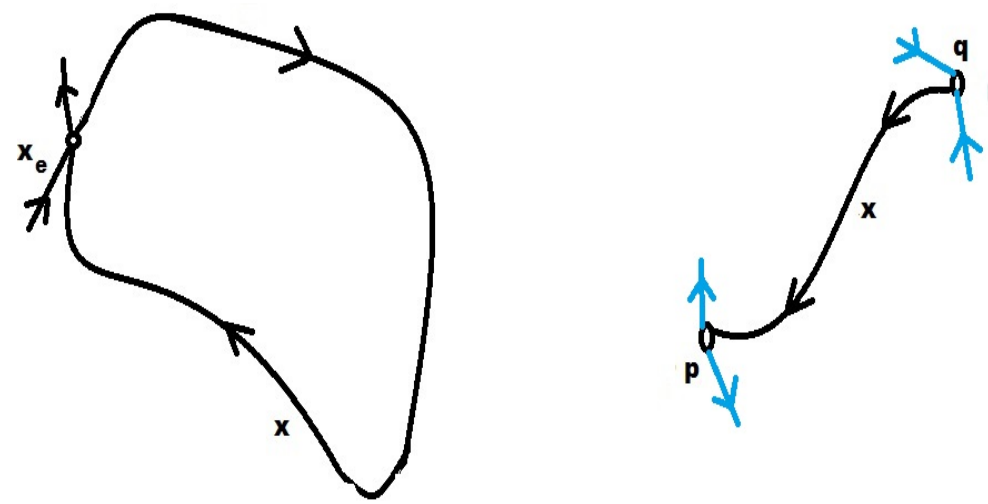

Figure 3: Left: Homoclinic orbit: $\omega(\mathbf{x}) = \mathbf{x}_e = \alpha(\mathbf{x})$. Right: Heteroclinic orbit: $\omega(\mathbf{x}) = \mathbf{p}$; $\alpha(\mathbf{x}) = \mathbf{q}$.