Cournot Duopoly with Linear Demand and Linear Costs

Let q_1 and q_2 be the quantities of homogeneous items produced by two firms with associated marginal costs c_1 and c_2 per item respectively.
Items sell at $P = a - b(q_1 + q_2)$ each and it is assumed that all items produced are sold.
The profits made by the firms are then

$$\pi_1 = Pq_1 - c_1 q_1 = (a - c_1 - b(q_1 + q_2))\, q_1$$

$$\pi_2 = Pq_2 - c_2 q_2 = (a - c_2 - b(q_1 + q_2))\, q_2$$

respectively.
Maximising π_1 with respect to q_1

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= a - c_1 - b(q_1 + q_2) - bq_1 \\ &\overset{set}{=} 0 \\ \Rightarrow q_1 &= \frac{a - c_1}{2b} - \frac{1}{2} q_2 \qquad (1) \end{aligned}$$

Similarly maximising π_2 with respect to q_2 yields

$$q_2 = \frac{a - c_2}{2b} - \frac{1}{2} q_1 \qquad (2)$$

Equations (1) and (2) are referred to as *Reaction Functions* - provided their solutions are nonnegative, which I'll assume in the following.
If we are outside the system we can solve equations (1) and (2) simultaneously to get

$$q_1 = q_1^* \triangleq \frac{a - 2c_1 + c_2}{3b}, \qquad q_2 = q_2^* \triangleq \frac{a - 2c_2 + c_1}{3b} \qquad (3)$$

the quantities each firm should produce at each production cycle. However, within the system, neither firm knows what the other is producing at the current cycle, hence it might be more realistic to use e.g. the discrete dynamical system

$$\begin{aligned} q_1(k+1) &= \frac{a - c_1}{2b} - \frac{1}{2} q_2(k) \qquad (4) \\ q_2(k+1) &= \frac{a - c_2}{2b} - \frac{1}{2} q_1(k) \qquad (5) \end{aligned}$$

as a production policy where $q_1(k), q_2(k)$ are the quantities produced at cycle k.
More risk averse companies might prefer to use e.g.

$$\begin{aligned} q_1(k+1) &= \alpha_1 q_1(k) + (1 - \alpha_1) \left(\frac{a - c_1}{2b} - \frac{1}{2} q_2(k) \right) \qquad (6) \\ q_2(k+1) &= \alpha_2 q_2(k) + (1 - \alpha_2) \left(\frac{a - c_2}{2b} - \frac{1}{2} q_1(k) \right) \qquad (7) \end{aligned}$$

where $0 \le \alpha_1, \alpha_2 \le 1$. This represents a weighted version of what was produced last time and what the *Cournot* rule says should be produced. Notice that when $\alpha_1, \alpha_2 = 0$, the risk averse system reduces to the original dynamical system.

Both the original and risk averse dynamical systems may be written as the linear system with input:

$$\begin{bmatrix} q_1(k+1) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} \alpha_1 & -\frac{(1-\alpha_1)}{2} \\ -\frac{(1-\alpha_2)}{2} & \alpha_2 \end{bmatrix} \begin{bmatrix} q_1(k) \\ q_2(k) \end{bmatrix} + \begin{bmatrix} (1-\alpha_1)\frac{a-c_1}{2b} \\ (1-\alpha_2)\frac{a-c_2}{2b} \end{bmatrix} \tag{8}$$

or, alternatively,

$$\begin{bmatrix} \delta q_1(k+1) \\ \delta q_2(k+1) \end{bmatrix} = \begin{bmatrix} \alpha_1 & -\frac{(1-\alpha_1)}{2} \\ -\frac{(1-\alpha_2)}{2} & \alpha_2 \end{bmatrix} \begin{bmatrix} \delta q_1(k) \\ \delta q_2(k) \end{bmatrix} \tag{9}$$

where $\delta q_1 = q_1 - q_1^*$, $\delta q_2 = q_2 - q_2^*$.

For the original system, show that the fixed point is $\begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix}$.

Cournot Duopoly with Isoelastic Demand and Linear Costs

If the price per item is $P(q_1, q_2) = \frac{d}{q_1+q_2}$ and the costs are as before, show that the reaction functions are given by

$$q_1 = \sqrt{\frac{d}{c_1} q_2} - q_2 \tag{10}$$

$$q_2 = \sqrt{\frac{d}{c_2} q_1} - q_1 \tag{11}$$

Assuming that these are well defined (i.e. result in nonnegative values), show that their simultaneous solution is given by

$$q_1 = 0, \qquad q_2 = 0$$

or

$$q_1 = \overline{q_1} \triangleq \frac{dc_2}{(c_1+c_2)^2}, \qquad q_2 = \overline{q_2} \triangleq \frac{dc_1}{(c_1+c_2)^2}. \tag{12}$$

As before, a discrete-time dynamical system may be defined

$$q_1(k+1) = \sqrt{\frac{d}{c_1} q_2(k)} - q_2(k) \tag{13}$$

$$q_2(k+1) = \sqrt{\frac{d}{c_2} q_1(k)} - q_1(k) \tag{14}$$

Both $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \overline{q_1} \\ \overline{q_2} \end{bmatrix}$ are fixed points of this system. This system is nonlinear, but its linearisation about $\begin{bmatrix} \overline{q_1} \\ \overline{q_2} \end{bmatrix}$ is

$$\begin{bmatrix} \delta q_1(k+1) \\ \delta q_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{-c_1+c_2}{2c_1} \\ \frac{c_1-c_2}{2c_2} & 0 \end{bmatrix} \begin{bmatrix} \delta q_1(k) \\ \delta q_2(k) \end{bmatrix} \tag{15}$$

where $\delta q_1 = q_1 - \overline{q_1}$, $\delta q_2 = q_2 - \overline{q_2}$.