1 Generalities: Flows & Maps

1. Linear Transformations
   The transformation $\mathbf{x} = T\mathbf{z}$ transforms

   (a) the linear flow $\dot{\mathbf{x}} = A\mathbf{x}$ to $\dot{\mathbf{z}} = T^{-1}AT\mathbf{z}$,

   (b) the linear map $\mathbf{x}' = A\mathbf{x}$ to $\mathbf{z}' = T^{-1}AT\mathbf{z}$,

   (c) the nonlinear flow

   $$\begin{aligned} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) &= A\mathbf{x} + \mathrm{O}\big(\mathbf{x}\big) \\ \text{to} \qquad \dot{\mathbf{z}} = \mathrm{T}^{-1}\mathbf{f}(\mathrm{T}\mathbf{z}) &= T^{-1}AT\mathbf{z} + T^{-1}\,\mathrm{O}\big(T\mathbf{z}\big) \end{aligned}$$

   (d) and the nonlinear map

   $$\begin{aligned} \mathbf{x}' = \mathbf{f}(\mathbf{x}) &= A\mathbf{x} + \mathrm{O}\big(\mathbf{x}\big) \\ \text{to} \qquad \mathbf{z}' = \mathrm{T}^{-1}\mathbf{f}(\mathrm{T}\mathbf{z}) &= T^{-1}AT\mathbf{z} + T^{-1}\,\mathrm{O}\big(T\mathbf{z}\big) \end{aligned}$$

   where $\mathrm{O}\big(.\big)$ are the nonlinear or higher order terms.
   In particular, if A is diagonalisable, then it is possible to choose

   $$T = E \overset{\triangle}{=} \big[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n\big]$$

   and

   $$T^{-1}AT = \Lambda \overset{\triangle}{=} \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

   where

   $$A\mathbf{e}_i = \lambda_i \mathbf{e}_i, \qquad i = 1, 2, \ldots, n.$$

2. Computing A^k and e^{At} when A is diagonalisable ($A = E\Lambda E^{-1}$)

   $$\begin{aligned} A^k &= E\Lambda^k E^{-1} \\ e^{At} &= Ee^{\Lambda t}E^{-1} \end{aligned}$$

3. The quadratic function

   $$q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j = \mathbf{x}^T P \mathbf{x},$$

   where $P = P^T$, is positive definite iff *the leading principal minors* of P are all positive (*Sylvester*'s criterion).

4. The characteristic polynomial of the $n \times n$ matrix A is $\chi(\lambda) = det(\lambda I - A)$. It may be computed by

$$\chi(\lambda) = \lambda^2 - tr\,\lambda + det, \qquad \text{if } n = 2$$

$$\chi(\lambda) = \lambda^3 - tr\,\lambda^2 + \left(\sum_{i=1}^{3} M_{ii}\right)\lambda - det, \qquad \text{if } n = 3$$

where tr, M_{ii} and det are the trace, the minor associated with position (i, i) and the determinant of A respectively.

5. *Hartman-Grobman* Theorem: The behaviour of a dynamical system in a neighbourhood of a hyperbolic fixed point is qualitatively the same as that of the linearised system. A hyperbolic fixed point is one whose *Jacobian* matrix has **no** eigenvalue with

   (a) real part equal to zero (flows),

   (b) modulus equal to one (maps).

2 Flows

6. Stability of flows and characteristic polynomials:

   (a) A fixed point of the 2-d flow with characteristic polynomial $\chi(\lambda) = \lambda^2 + a_1\lambda + a_0$ is stable iff

   $$a_1 > 0, \qquad a_0 > 0.$$

   (b) A fixed point of the 3-d flow with characteristic polynomial $\chi(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ is stable iff

   $$a_2 > 0,\ a_1 > 0,\ a_0 > 0, \qquad a_2 a_1 > a_0.$$

7. Invariant manifolds of 2-d flows: If the flow

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

with fixed point at the origin has an associated invariant manifold $y = h(x)$, then h is the solution of

$$g(x, h(x)) = Dh(x) f(x, h(x)), \qquad h(0) = 0$$

8. *Lyapunov*'s Direct Method for flows (*Lyapunov* Functions): For the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

the origin is

   (a) locally stable if there exists a positive definite function $V(\mathbf{x})$ with negative semi-definite time derivative $\dot{V}$,

   (b) locally asymptotically stable if in addition $\dot{V}$ is negative definite (strict *Lyapunov* function).

The origin is globally asymptotically stable if V is a strict *Lyapunov* function for which $V \to \infty$ as $||\mathbf{x}|| \to \infty$ (V is radially unbounded).

9. *Lyapunov*'s Direct Method for Linear flows: The origin of the system $\dot{\mathbf{x}} = A\mathbf{x}$ is globally asymptotically stable if for any real symmetric positive definite Q, the solution P of the *Lyapunov* Equation

$$A^T P + PA = -Q$$

is also real symmetric positive definite.

10. A modified *LaSalle* Invariance Principle for flows: For the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i) V is positive definite, (ii) $\frac{dV}{dt}$ is negative semi-definite and (iii) the union of all invariant sets with $\frac{dV}{dt} = 0$ is $M = \{\mathbf{0}\}$, then $\mathbf{0}$ is asymptotically stable. In particular for the system

$$\dot{x} = y, \qquad \dot{y} = -c(x) - b(y),$$

where b and c are continuous functions with the same sign as their arguments, the origin $\mathbf{0}$ is asymptotically stable. This may be shown using

$$V(\mathbf{x}) = \frac{1}{2}x_2^2 + \int_0^{x_1} c(s)\, ds.$$

11. Index Theory:

    (a) The index of a node or focus is $+1$, of a saddle is -1.

    (b) Every periodic orbit encloses fixed points whose indices sum to $+1$.

12. *Dulac*'s criterion: Consider the 2-d continuously differentiable flow $\dot{x} = f(x,y),\ \dot{y} = g(x,y)$. If there exists a continuously differentiable function $\beta(x,y)$ such that on a simply connected region R_0,

$$\frac{\partial}{\partial x}\left[\beta(x,y)f(x,y)\right] + \frac{\partial}{\partial y}\left[\beta(x,y)g(x,y)\right]$$

is unchanged in sign, then there are **no** periodic orbits contained in R_0.

13. *Poincaré - Bendixson* theorem: Consider the 2-d continuously differentiable flow $\dot{x} = f(x,y),\ \dot{y} = g(x,y)$. If R_1 is a closed bounded trapping region containing no fixed points, then there exists a periodic orbit in R_1.

14. Cartesian-Polar transformation:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \arctan\left(\tfrac{y}{x}\right) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

15. Method of Computation of a Periodic Orbit for a particular class of 2-d flows using a *Poincaré* Map : For the flow described by

$$\begin{aligned} \dot{r} &= F(r,\theta), \qquad & r(0) = r_0 \\ \dot{\theta} &= G(\theta), \qquad & \theta(0) = 0, \end{aligned}$$

writing the solution of this initial value problem as $(R(t,r_0),\ \Theta(t))$, then

(a) the period (T) of the orbit satisfies $\Theta(T) = 2\pi$,

(b) the *Poincaré* Map is given by $r_1 = P(r_0) \triangleq R(T, r_0)$.

The periodic orbit is the fixed point of the *Poincaré* Map, i.e. the solution r_e of

$$r_e = P(r_e) = R(T, r_e).$$

The stability of r_e is determined by the absolute value of $P'(r_e)$.
With $s(t, r_0) \triangleq \frac{\partial R}{\partial r_0}(t, r_0)$, s can be computed as the solution of

$$\dot{s} = \frac{\partial F}{\partial r}(r, \theta)s, \qquad s(0, r_0) = 1.$$

Then $P'(r_e) = s(T, r_e)$.

16. The 1-d flow $\dot{x} = f(x, a)$ has a fixed point x_e which depends on the parameter a; x_e has a bifurcation at $a = a_c$ if $Df(x_e, a_c) = 0$.

17. *Hopf* Bifurcation: The 2-d flow $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a)$ has a fixed point $\mathbf{x}_e$ which depends on the parameter a. If the eigenvalues may be written as

$$\lambda(a) = \Re(a) \pm i\Im(a)$$

and for which, at $a = a_c$,

$$\Re(a_c) = 0, \qquad \Im(a_c) \neq 0, \qquad \frac{d\Re}{da}(a_c) \neq 0$$

then $\mathbf{x}_e$ has a *Hopf* Bifurcation at $a = a_c$.

3 Maps

18. Stability of 2-d maps and characteristic polynomials:
A fixed point of the 2-d map with characteristic polynomial
$\chi(\lambda) = \lambda^2 + a_1\lambda + a_0$ is stable iff

$$|a_1| < 1 + a_0, \qquad |a_0| < 1.$$

19. Invariant manifolds of 2-d maps: If the map

$$x' = f(x, y), \qquad y' = g(x, y)$$

with fixed point at the origin has an associated invariant manifold $y = h(x)$, then h is the solution of

$$g(x, h(x)) = h\left(f(x, h(x))\right), \qquad h(0) = 0$$

20. *Lyapunov*'s Direct Method for maps (*Lyapunov* Functions): For the system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

the origin is

(a) locally stable if there exists a positive definite function $V(\mathbf{x})$ with negative semi-definite $\Delta V \triangleq V(\mathbf{x}') - V(\mathbf{x})$,

(b) locally asymptotically stable if in addition ΔV is negative definite (strict *Lyapunov* function).

The origin is globally asymptotically stable if V is a strict *Lyapunov* function for which $V \to \infty$ as $||\mathbf{x}|| \to \infty$ (V is radially unbounded).

21. *Lyapunov*'s Direct Method for Linear maps: The origin of the system $\mathbf{x}' = A\mathbf{x}$ is globally asymptotically stable if for any real symmetric positive definite Q, the solution P of the Discrete *Lyapunov* Equation

$$A^T PA - P = -Q$$

is also real symmetric positive definite.

22. The 1-d map $x' = f(x, a)$ has a fixed point x_e which depends on the parameter a; x_e has a bifurcation at $a = a_c$ if $|Df(x_e, a_c)| = 1$.

23. *Neimark-Sacker* Bifurcation:The 2-d map $\mathbf{x}' = \mathbf{f}(\mathbf{x}, a)$ has a fixed point $\mathbf{x}_e$ which depends on the parameter a. If the eigenvalues may be written as

$$\lambda(a) = R(a)e^{\pm i\theta(a)}$$

and for which, at $a = a_c$,

$$R(a_c) = 1, \qquad 0 < \theta(a_c) < \pi, \qquad \frac{dR}{da}(a_c) \neq 0, \qquad e^{\pm ik\theta(a_c)} \neq 1, \;\; k = 1, 2, 3, 4,$$

then $\mathbf{x}_e$ has a *Neimark-Sacker* Bifurcation at $a = a_c$.