What's "Dynamical Systems" all about?

The subject known as Dynamical Systems (DS) is also called Nonlinear Dynamics and sometimes Chaos Theory.

For the autonomous system described by the differential equation

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{f}(\mathbf{x}(t)), \qquad 0 \le t < \infty \tag{1}$$

let $\mathbf{\Phi}(t, \mathbf{x}_0)$ represent its solution starting in the initial state $\mathbf{x}(0) = \mathbf{x}_0$. Such a solution curve is called a *trajectory* or *orbit*. We assume that $\mathbf{f}$ is such that $\mathbf{\Phi}(t, \mathbf{x}_0)$ exists and is unique. We'll also shorten Equation (1) to read

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2}$$

Similarly, for the autonomous system described by the difference equation

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k)), \qquad k = 0, 1, 2, \ldots \tag{3}$$

let $\mathbf{\Phi}(k, \mathbf{x}_0)$ represent its solution/trajectory/orbit starting from $\mathbf{x}(0) = \mathbf{x}_0$. In fact $\mathbf{\Phi}(k, \mathbf{x}_0) = \mathbf{f}^k(\mathbf{x}_0)$ is the k-fold composition of $\mathbf{f}$ with itself acting on $\mathbf{x}_0$. Again we'll shorten Equation (3) to read

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{4}$$

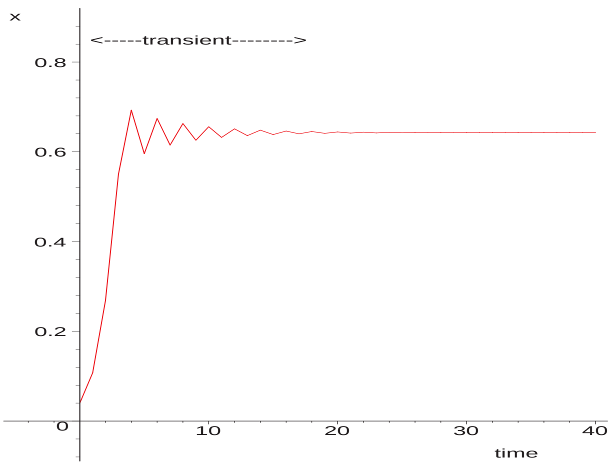

Figure 1: System exhibits an equilibrium or fixed state

Some typical orbits are shown in Figures 1-3. A feature of these orbits is that after an initial transient phase, the trajectory typically settles down to its "long-term" behaviour, be this regular or otherwise. Regular behaviours include being in a state of equilibrium or exhibiting periodic motion. The otherwise includes chaotic motion.

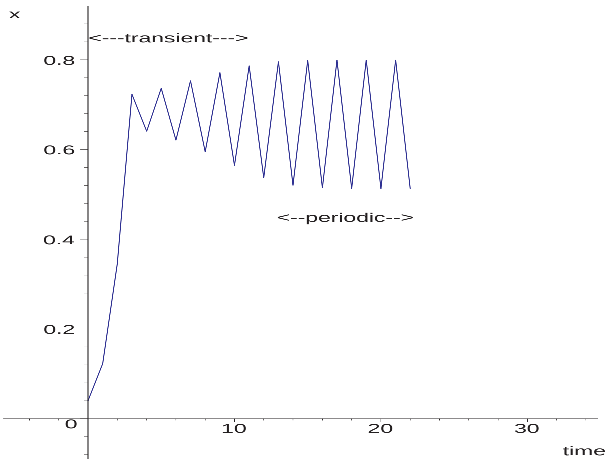

Figure 2: System exhibits periodic behaviour

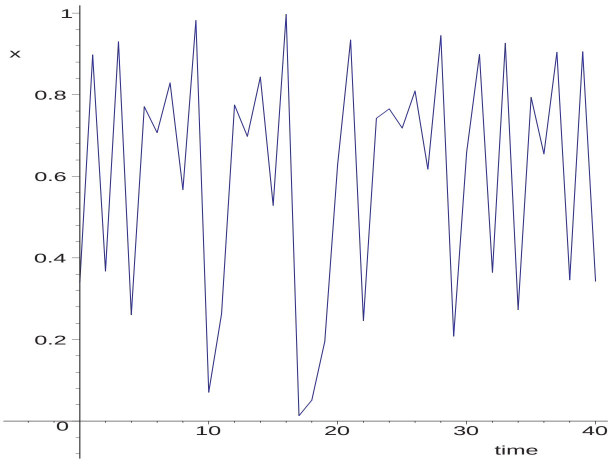

Figure 3: System exhibits chaotic behaviour

One of the principal concerns of DS is to describe this long-term behaviour; to this end we introduce the following definitions. Let τ stand for t or k as appropriate. A point z is an ω-limit point of (the orbit of) $\mathbf{x}_0$ if $\mathbf{\Phi}(\tau, \mathbf{x}_0)$ repeatedly comes close to z as τ tends to infinity, i.e. there exists a sequence of time values $\tau_j,\ j \in \mathbf{N}$ such that $\mathbf{\Phi}(\tau_j, \mathbf{x}_0) \to z$ as $\tau_j \to \infty$. The ω-limit set of $\mathbf{x}_0$, denoted $\omega(\mathbf{x}_0)$, is the set of ω-limit points of $\mathbf{x}_0$[1].

With respect to the orbits shown in Figures 1-3, the omega-limit sets are respectively seen to be

(a) $\omega(\mathbf{x}_0) = \{0.6429\}$

(b) $\omega(\mathbf{x}_0) = [0.5130,\ 0.7995]$

(c) $\omega(\mathbf{x}_0) = (0, 1)$

[1]We shall occasionally have recourse to $\alpha(\mathbf{x}_0)$, the α-limit set of $\mathbf{x}_0$ which is defined akin to $\omega(\mathbf{x}_0)$ except that $\tau \to -\infty$

Another of the concerns of DS is to describe not only the long-term behaviours seen in the $\omega(\mathbf{x}_0)$ associated with typical orbits, but also all other behaviours that the system is capable of, and to discover the boundaries between regions of the state space with different behaviours. This involves identifying

(1) the equilibria or fixed states,

(2) the periodic orbits,

(3) and the invariant sets (a set is invariant if an orbit which originates in the set always remains in the set)

among other dynamical features. (Notice that the definition of invariant set given above includes both fixed points and periodic orbits).

Another aim of DS is to identify which dynamical features are stable or attracting - roughly speaking which features attract nearby orbits, and to determine whether these features are structurally stable - impervious to small changes in parameter values - by means of Bifurcation Analysis.

Although we have talked about orbits and trajectories, DS does not "solve" the underlying differential or difference equations in the sense of computing $\mathbf{\Phi}(\tau, \mathbf{x}_0)$ as a function of τ and $\mathbf{x}_0$, but rather it draws its conclusions about dynamical features and their stability properties from an analysis of the structure of the right hand sides of Equations (2) and (4).