Cournot Duopoly with Linear Demand and Linear Costs

Let q_1 and q_2 be the quantities of homogeneous items produced by two firms with associated marginal costs c_1 and c_2 per item respectively.
Items sell at $P = a - b(q_1 + q_2)$ each and it is assumed that all items produced are sold.
The profits made by the firms are then

$$\pi_1 = Pq_1 - c_1q_1 = (a - c_1 - b(q_1 + q_2))\, q_1$$

$$\pi_2 = Pq_2 - c_2q_2 = (a - c_2 - b(q_1 + q_2))\, q_2$$

respectively.
Maximising π_1 with respect to q_1

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= a - c_1 - b(q_1 + q_2) - bq_1 \\ &\overset{set}{=} 0 \\ \Rightarrow q_1 &= \frac{a - c_1}{2b} - \frac{1}{2}q_2 \qquad (1) \end{aligned}$$

Similarly maximising π_2 with respect to q_2 yields

$$q_2 = \frac{a - c_2}{2b} - \frac{1}{2}q_1 \qquad (2)$$

Equations 1 and 2 are referred to as *Reaction Functions* - provided their solutions are nonnegative, which I'll assume in the following.
Solving equations 1 and 2 simultaneously gives the *equilibrium* values

$$q_1^* = \frac{a - 2c_1 + c_2}{3b}, \qquad q_2^* = \frac{a - 2c_2 + c_1}{3b}$$

At these equilibrium values

$$P^* = \frac{a + c_1 + c_2}{3}$$

and

$$\pi_1^* = \frac{(a - 2c_1 + c_2)^2}{9b}, \qquad \pi_2^* = \frac{(a - 2c_2 + c_1)^2}{9b} \qquad (3)$$

Cournot duopoly is an example of a 2-player matrix form game with an infinite number of strategies available to both players (firms), i.e. the choice of q_1 and q_2 respectively. $\langle q_1^*, q_2^* \rangle$ is then a *Nash* equilibrium with payoffs π_1^* and π_2^* respectively.

Stackelberg Duopoly

Stackelberg duopoly is an example of a 2-player extensive form game in which Firm 1 moves first (the "Leader") and Firm 2 responds (the "Follower"). Irrespective of what the leader does, the follower will use the reaction function (Eq. 2) as it is its best response.

Knowing this, the leader seeks to maximise

$$\Pi_1 = \left(a - c_1 - b\left(q_1 + \frac{a - c_2}{2b} - \frac{1}{2}q_1\right)\right) q_1 = \left(a - c_1 - b\left(\frac{q_1}{2} + \frac{a - c_2}{2b}\right)\right) q_1$$

as a function of q_1.

$$\begin{aligned} \frac{\partial \Pi_1}{\partial q_1} &= a - c_1 - b\left(\frac{q_1}{2} + \frac{a - c_2}{2b}\right) - b\frac{q_1}{2} \\ &\overset{set}{=} 0 \\ \Rightarrow q_1 &= \frac{a - 2c_1 + c_2}{2b} \end{aligned} \tag{4}$$

Denoting this optimal value by Q_1^* and the corresponding value of q_2 by Q_2^* (substitute Eq. 4 into Eq. 2) gives

$$Q_1^* = \frac{a - 2c_1 + c_2}{2b}, \qquad Q_2^* = \frac{a + 2c_1 - 3c_2}{4b}$$

At these equilibrium values

$$P^* = \frac{a + 2c_1 + c_2}{4}$$

and

$$\Pi_1^* = \frac{(a - 2c_1 + c_2)^2}{8b}, \qquad \Pi_2^* = \frac{(a + 2c_1 - 3c_2)^2}{16b} \tag{5}$$

Comparing Cournot & Stackelberg Duopoly Games

From Eqs 3 and 5,

$$\pi_1^* < \Pi_1^*$$

for all parameter values, but it can be shown that

$$\pi_2^* > \Pi_2^*$$

whenever $a - 2c_1 + c_2 > 0$. This corresponds to $Q_1^* > 0$.
Exercise: prove the second assertion.