Controlling Chaos: OGY control

Recall that an attractor is a set in state space towards which trajectories converge with increasing time. For a chaotic system, a chaotic attractor has the properties that

(i) embedded in the attractor is an infinite number of unstable periodic orbits (UPOs), e.g. fixed points, period-2 points, etc.,

(ii) if the attractor is ergodic then a generic trajectory will eventually come arbitrarily close to any one of these orbits,

(iii) it has exponential sensitivity to small perturbations.

Hence it seems that the goal of stabilising a (pre-selected) UPO can potentially be achieved by small parameter perturbations. This is one type of "control of chaos". Here by control is meant feedback control: the control is chosen as a function of the state value.
We shall illustrate the approach by concentrating on the *Ott, Grebogi* & *Yorke* (OGY) method, which is a technique for chaotic maps. The method was initially developed in the context of time series measurements only being available for the system. We shall present it assuming that a model of the system is available.

The system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, p) \tag{1}$$

is chaotic for the nominal value of the parameter $p = p_0$. It is intended to stabilise the corresponding fixed point $\mathbf{x}_e$ (this is the pre-selected UPO being used here), by means of OGY control, i.e. in a small neighbourhood of the fixed point the trajectory of the system can be actively controlled using an allowably perturbed value of the parameter, while outside this neighbourhood the system evolves using the nominal dynamics only. Symbolically, this can be stated as: if ϵ is the width of the neighbourhood in which active control takes place and Δp is the maximum allowable parameter variation , then p is chosen so that

$$\begin{cases} p_0 - \Delta p \le p \le p_0 + \Delta p, & \text{if } ||\mathbf{x} - \mathbf{x}_e|| \le \epsilon, \\ p = p_0, & \text{otherwise} \end{cases}$$

Furthermore, we design the control strategy to satisfy

$$p - p_0 = K(\mathbf{x} - \mathbf{x}_e), \qquad \text{if } ||\mathbf{x} - \mathbf{x}_e|| \le \epsilon \tag{2}$$

where K is a constant $1 \times n$ vector whose value is to be chosen to achieve the control goal. How is K chosen? One way is as follows. Since $||\mathbf{x} - \mathbf{x}_e||$ and $|p - p_0|$ are small in the active neighbourhood, we can approximate the system by its linearisation - expand $\mathbf{f}$ about $(\mathbf{x}_e, p_0)$ to get

$$\begin{aligned} (\mathbf{x} - \mathbf{x}_e)' = \mathbf{x}' - \mathbf{x}_e &= \mathbf{f}(\mathbf{x}, p) - \mathbf{f}(\mathbf{x}_e, p_0) \\ &= A(\mathbf{x} - \mathbf{x}_e) + B(p - p_0) + \underbrace{\text{h.o.t.}}_{neglect} \end{aligned}$$

where

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_e, p_0), \qquad B = \frac{\partial \mathbf{f}}{\partial p}(\mathbf{x}_e, p_0) \tag{3}$$

With the control strategy $p - p_0 = K(\mathbf{x} - \mathbf{x}_e)$, the linearised system in the active neighbourhood becomes

$$(\mathbf{x}-\mathbf{x}_e)' = A(\mathbf{x}-\mathbf{x}_e) + BK(\mathbf{x}-\mathbf{x}_e) = (A+BK)(\mathbf{x}-\mathbf{x}_e)$$

which has solution

$$\mathbf{x}-\mathbf{x}_e = (A+BK)^k(\mathbf{x}_0-\mathbf{x}_e)$$

If K can be chosen so that all the eigenvalues of $A+BK$ have magnitude less than 1, i.e.

$$\text{spectral radius } (A+BK) \triangleq \max_{1\leq i\leq n} |\lambda_i| < 1 \tag{4}$$

then $\mathbf{x}-\mathbf{x}_e \to \mathbf{0}$ as $k \to \infty$, which is the desired goal. Generically, it is usually possible to choose an appropriate K vector.
Notice that this approach constrains the values for ϵ, Δp and K. To ensure that all parameter perturbations are allowable when in the active neighbourhood, it is necessary to have

$$||K||\,\epsilon \leq \Delta p \tag{5}$$

In summary, if K, ϵ and Δp are chosen to satisfy Eqs (4) and (5), then the control strategy applied to the system (Eq (1)) is to chose

$$p = \begin{cases} p_0 + K(\mathbf{x}-\mathbf{x}_e), & \text{if } ||\mathbf{x}-\mathbf{x}_e|| \leq \epsilon, \\ p_0, & \text{otherwise} \end{cases} \tag{6}$$

Fig. 1 contrasts a typical trajectory entering the neighbourhood of the target ($\mathbf{x}_e$) with and without control.

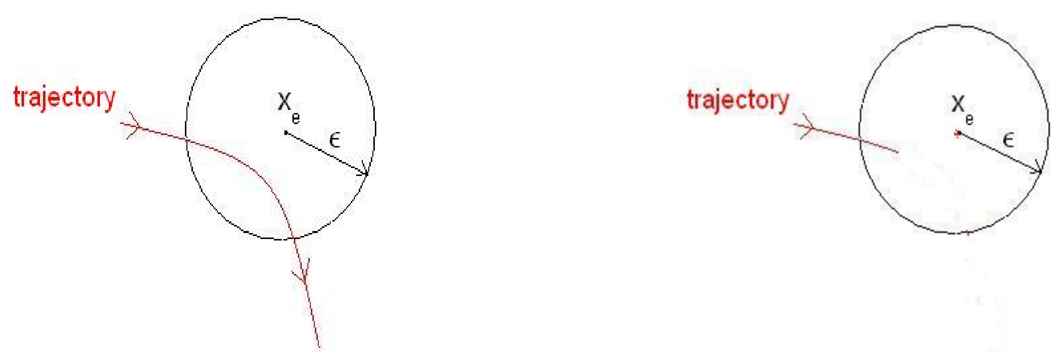

Figure 1: Trajectory entering active neighbourhood: (left) without control, (right) with OGY control - the parameter perturbation "kicks" the trajectory onto the target.

Example: Logistic map
The map

$$x' = f(x,a) = ax(1-x), \qquad a_0 = 3.9$$

is chaotic for the nominal value $a_0 = 3.9$. It has a corresponding fixed point at

$$x_e = 1 - \frac{1}{a_0} \approx 0.7439.$$

If we are told that the maximum allowable parameter perturbation is $\Delta a = 1\% = 0.039$, find an OGY controller that makes x_e superstable when in the active region.
We compute

$$A = \frac{\partial f}{\partial x}(x_e, a_0) = a_0(1 - 2x_e) = -1.9, \qquad B = \frac{\partial f}{\partial a}(x_e, a_0) = x_e(1 - x_e) \approx 0.1907$$

In this 1-d case, K is a scalar. To make x_e superstable, the eigenvalue of $A + BK = 0$ i.e. K is chosen as $K = -\frac{A}{B} \approx 9.965$. To satisfy Eq (5), we must choose $\epsilon < 0.003914$. Thus the control strategy is

$$a = \begin{cases} 3.9 + 9.965(\mathbf{x} - 0.7439), & \text{if } |\mathbf{x} - 0.7439| \leq 0.003914, \\ 3.9, & \text{otherwise} \end{cases}$$

Fig. 2 shows a typical trajectory for this controlled system.

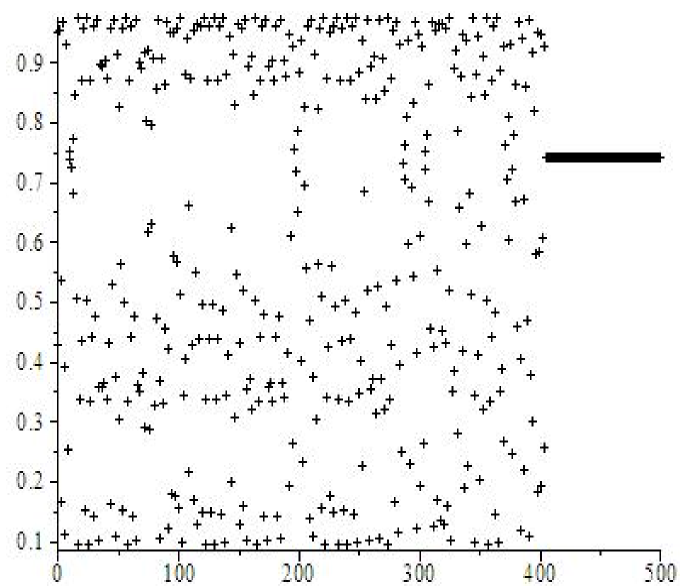

Figure 2: The trajectory of the Logistic Map with OGY control starting from $x_0 = 0.43$