Characterisation of Planar Flows

Consider a planar (2-d) flow

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

with fixed point at $(\overline{x},\ \overline{y})$ which has the associated characteristic equation

$$\lambda^2 - a\lambda + b = 0$$

where $a = \left(\frac{\partial f}{\partial x}(\overline{x},\overline{y}) + \frac{\partial g}{\partial y}(\overline{x},\overline{y})\right)$ and $b = \frac{\partial f}{\partial x}(\overline{x},\overline{y})\frac{\partial g}{\partial y}(\overline{x},\overline{y}) - \frac{\partial f}{\partial y}(\overline{x},\overline{y})\frac{\partial g}{\partial x}(\overline{x},\overline{y})$.
The fixed point is classified as a

stable node or focus	when	$a < 0$, $b > 0$
completely unstable node or focus	when	$a > 0$, $b > 0$
saddle	when	$b < 0$

Furthermore, the fixed point is a focus if $a^2 < 4b$, and a node otherwise.

Summarising (see Fig. 1 also)

Classification of $\mathbf{x}_e = \mathbf{0}$	Relationship between a and b
stable node (sink)	$a < 0,\ b > 0,\ (a/2)^2 > b$
unstable node (source)	$a > 0,\ b > 0,\ (a/2)^2 > b$
saddle	$b < 0$
stable focus (sink)	$a < 0,\ b > 0,\ (a/2)^2 < b$
unstable focus (source)	$a > 0,\ b > 0,\ (a/2)^2 < b$

Note: $a = 0,\ b > 0$ corresponds to the non-hyperbolic (degenerate) case $\lambda_{1,2} = \pm i\sqrt{b}$, which would indicate a *centre* in the linear case, while $b = 0$ corresponds to the non-hyperbolic case $\lambda_1 = 0,\ \lambda_2 = -a$.

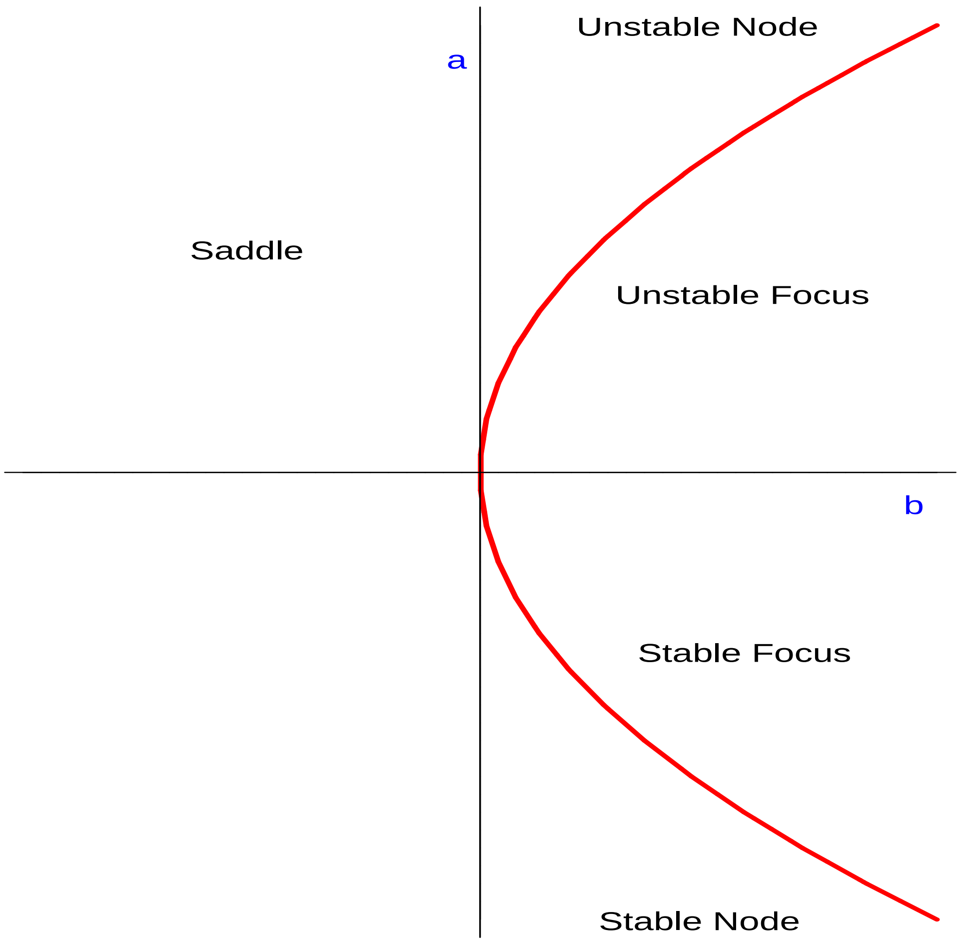

Figure 1: Stability in $a-b$ plane.