The Two-Dimensional Sobolev Inequality in the Case of an Arbitrary Grid

N. V. Kopteva

Department of Computational Mathematics and Cybernetics, Moscow State University,
Vorob'evy gory, Moscow, 119899 Russia

Received November 26, 1996; in final form, June 16, 1997

Abstract—It is found that, for piecewise-linear continuous functions given on the triangulation \( T_h \) of a domain \( \Omega \subset \mathbb{R}^2 \) with a piecewise-smooth boundary, the norms in \( C(\Omega) \) are bounded by the norms in \( W^1_2(\Omega) \) multiplied by \( c \| \ln h \|_{1/2} \), where \( h \) is the smallest diameter of the triangles \( \tau \in T_h \) and \( c = c(\Omega) \) is a constant. Here, it is not assumed that the triangulation \( T_h \) is quasi-continuous.

It is known [1] that, if \( \Omega = (a, b) \), then \( W^1_2(\Omega) \subset C(\Omega) \) and

\[ \| v \|_{C(\Omega)} \leq c \| v \|_{W^1_2(\Omega)}, \]

for any function \( v \in W^1_2(\Omega) \), where \( c = c(\Omega) \) is a constant independent of \( v \). If \( \Omega \subset \mathbb{R}^2 \), the embedding (1) is not valid. However, if we introduce the triangulation

\[ T_h: \Omega = \bigcup_{\tau \in T_h} \tau \]

in \( \Omega \) and assume that this triangulation is quasi-continuous (i.e., that the diameters of all triangles \( \tau \) do not exceed \( h \) and their areas are no less than \( ch^2 \), where \( c > 0 \) is a constant independent of \( h \)), then piecewise-linear continuous functions defined on this triangulation satisfy inequality (1) with the constant \( c = c(h) = \bar{c} \| \ln h \|_{1/2} \) (see [2–4]):

\[ \| v \|_{C(\Omega)} \leq \bar{c} \| \ln h \|_{1/2} \| v \|_{W^1_2(\Omega)}. \]

The aim of this work is to establish an inequality similar to (2) for the triangulations that, generally speaking, are not quasi-continuous and whose elements may be triangles of arbitrary form.

Suppose that \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with a piecewise-smooth boundary whose angles are different from 0 and \( 2\pi \) and \( T_h \) is the triangulation of this domain; i.e., \( \Omega = \bigcup_{\tau \in T_h} \tau \) and

\[ h = \min_{\tau \in T_h} \text{diam} \tau. \]

The following theorem is a fundamental result of this work.

**Theorem 1.** Let \( \chi \) be a continuous function defined on \( \Omega \). Suppose that this function is linear on each triangle \( \tau \) of \( T_h \). Then,

\[ \| \chi \|_{C(\Omega)} \leq c \| \chi \|_{W^1_2(\Omega)}, \]

where \( h \) is a parameter of triangulation \( T_h \) defined by (3) and \( c = c(\Omega) \) is a constant independent of \( \chi \) and \( T_h \).

Since, for the functions from \( W^1_2(\Omega) \) that vanish on a part of finite length of the boundary \( \Omega \), the seminorm \( \| v \|_{W^1_2(\Omega)} \equiv \| \nabla v \|_{L^2(\Omega)} \) is equivalent to the norm \( \| v \|_{W^1_2(\Omega)} \) (see [5]), Theorem 1 implies the following statement.
Corollary. Suppose that the hypotheses of Theorem 1 are valid and \( \chi = 0 \) on \( S \subseteq \partial \Omega \), where \( S \) is part of the boundary of finite length. Then,

\[
\|\chi\|_{C(\Omega)} \leq c \sqrt{\ln h + 1} \|\nabla \chi\|_{L^2(\Omega)},
\]

where \( c = c(\Omega, S) \) is a constant independent of \( \chi \).

Remark 1. If we define the grid analogues of the norms of \( W^1 \) and \( C \) for the grid functions defined at the vertices of the triangles \( t \) (see [3, 6, 7]), then it is clear that a theorem analogous to Theorem 1 is valid.

Theorem 1 is a corollary to the more general Theorem 2.

Theorem 2. Suppose that \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with a piecewise-smooth boundary whose angles are different from 0 and \( 2\pi \) and the function \( \chi \in W^1(\Omega) \) is such that its restriction onto a triangle \( t \subset \Omega \) is a linear function. Then,

\[
\|\chi\|_{C(t)} \leq c_1 \left( \frac{\ln c_2}{\text{diam} t} \right)^{1/2} \|\chi\|_{W^1(\Omega)},
\]

where \( c_1 = c_1(\Omega) \) and \( c_2 = c_2(\Omega) \) are constants independent of \( \chi \) and \( t \).

Indeed, under the hypotheses of Theorem 1, we obtain, in view of Theorem 2, that

\[
\|\chi\|_{C(t)} \leq c_1 \max_{t \in \mathcal{T}} \left( \frac{\ln c_2}{\text{diam} t} \right)^{1/2} \|\chi\|_{W^1(\Omega)}.
\]

Hence, we obtain the assertion of Theorem 1 with the constant \( c = c_1 \max\{1, |\ln c_2|^{1/2}\} \).

The following lemma will be useful for proving Theorem 2.

Lemma. If \( \chi \) is a linear function defined on a triangle \( t \), then

\[
\|\chi\|_{C(t)} \leq c_3 (\text{mes} t)^{-1/2} \|\chi\|_{L^2(t)},
\]

where \( c_3 = 1/3 \).

Proof. Denoting by \( a, b, \) and \( c \) the values of the function \( \chi \) at the vertices of the triangle \( t \) and taking into account that the function \( \chi \) is linear, we obtain \( \|\chi\|_{C(t)} = \max\{|a|, |b|, |c|\} \). Then, the relations

\[
\|\chi\|_{L^2(t)}^2 = \frac{\text{mes} t}{6} (a^2 + b^2 + c^2 + ab + ac + bc) = \frac{\text{mes} t}{6} \left[ (a+b/2+c/2)^2 + \frac{3}{4}(b+c/3)^2 + \frac{3}{4}c^2 \right]
\]

yield the assertion of the lemma.

Proof of Theorem 2. Let \( B \) be an open circle of radius \( R < \text{diam} \Omega \) such that \( \overline{\Omega} \subset B \). It is known [1, 3] that \( \chi \in W^1(\Omega) \) can be continued to \( B \) with its norm preserved, i.e.,

\[
\|\chi\|_{W^1(B)} \leq c_4 \|\chi\|_{W^1(\Omega)},
\]

so that \( \chi|_{\partial B} = 0 \); here, \( c_4 = c_4(\Omega, B) \) is a constant independent of \( \chi \).

Since \( \chi \) is linear in \( t \), then, by virtue of the lemma,

\[
\|\chi\|_{C(t)} \leq c_3 (\text{mes} t)^{-1/2} \|\chi\|_{L^2(t)}.
\]

It is clear that

\[
\|\chi\|_{L^2(t)} = (\chi, \phi),
\]

where

\[
(\mu, \nu) = \int_B \mu(M) \nu(M) dM.
\]
is a scalar product in \( L_2(B) \) and
\[
\phi(M) = \begin{cases} \chi/\|\chi\|_{L_2(\tau)}, & M \in \tau, \\ 0, & M \notin \tau, \end{cases}
\]
is a function from \( L_2(B) \), where
\[
\|\phi\|_{L_2(B)} = 1.
\]

Let us introduce a function \( \nu \in \hat{W}^{-1}_2(B) \), which is a generalized solution to the problem
\[
-\Delta \nu(M) = \phi(M), \quad M \in B, \quad \nu(M) = 0, \quad M \in \partial B;
\]
i.e.,
\[
\nu \in \hat{W}^{-1}_2(B): (\nabla \nu, \nabla \varphi) = (\phi, \varphi) \quad \forall \varphi \in \hat{W}^{1}_2(B).
\]
Then, taking into account (6),
\[
\|\chi\|_{L_2(\tau)} = (\nabla \nu, \nabla \chi) \leq \|\nabla \nu\|_{L_2(B)} \|\chi\|_{W^{-1}_2(B)},
\]
and, by virtue of (4),
\[
\|\chi\|_{L_2(\tau)} \leq C_4 \|\nabla \nu\|_{L_2(B)} \|\chi\|_{W^{-1}_2(\Omega)}.
\]

Next, noting that
\[
\|\nabla \nu\|_{L_2(B)} = \sqrt{(\nu, \phi)} \leq \|\nu\|_{L_2(\tau)},
\]
by virtue of (10), (7), and (8), we arrive at the inequality
\[
\|\chi\|_{L_2(\tau)} \leq C_4 \sqrt{\|\nu\|_{L_2(\tau)} \|\chi\|_{W^{-1}_2(\Omega)}},
\]
which, in view of (5), yields
\[
\|\chi\|_{C(\tau)} \leq C_3 c_4 (\text{mes } \tau)^{-1/2} \sqrt{\|\nu\|_{L_2(\tau)} \|\chi\|_{W^{-1}_2(\Omega)}},
\]
To estimate \( \sqrt{\|\nu\|_{L_2(\tau)}} \), we take advantage of the following representation of the function \( \nu(M) \) (see [8, p. 326]):
\[
\nu(M) = (G(M, P), \phi(P)).
\]
Denote by \( G(M, P) \) the Green's function of problem (9):
\[
G(M, P) = G(r, \psi; \rho, \theta) = \frac{1}{4\pi} \ln \frac{R^2 - 2r \rho \cos(\psi - \theta) + r^2 + \rho^2}{r^2 - 2r \rho \cos(\psi - \theta) + \rho^2} \leq \frac{1}{2\pi} \ln \frac{2R}{r_{MP}},
\]
where \( (r, \psi) \) and \( (\rho, \theta) \) are the polar coordinates of the points \( M \) and \( P \), respectively, and \( r_{MP} = \sqrt{r^2 + \rho^2 - 2r \rho \cos(\psi - \theta)} \) is the distance between \( M \) and \( P \). Hence,
\[
\|\nu\|_{L_2(\tau)}^2 = \int_{\tau} \nu^2(M) dM = \int_{\tau} (G(M, P), G(M, P)\phi(P))^2 dM,
\]
and, by virtue of the Cauchy–Schwarz–Bunyakovskii inequality,
\[
\|\nu\|_{L_2(\tau)}^2 \leq \left( \int_{\tau} G(M, Q) dQ \right) \left( \int_{\tau} G(M, P) \phi^2(P) dP \right) dM = \int_{\tau} \phi^2(P) \left( \int_{\tau} G(M, P) G(M, Q) dQ \right) dM dP.
\]
Taking into account (8), we obtain
\[
\|\nu\|_{L_2(\tau)}^2 \leq \max_{M \in B} \left( \int_{\tau} G(M, P) dP \right)^2.
\]
Let us denote by \( l \) the longest side of the triangle \( \tau \) and by \( d \), the altitude drawn to the side of length \( l \). Then,
taking into account that $(\sqrt{3}/2)diam \tau \leq l \leq diam \tau$, we obtain

$$
\|v\|_{L^1(\tau)} \leq \max_{M \in \mathcal{B}} \frac{d^{d/2}}{2\pi^{d/2}} \int_G \frac{G(M, P) dP}{\sqrt{x^2 + y^2}} \leq \frac{d}{\pi^{d/2}} \int_0^{2R} \frac{\ln \frac{2R}{r}}{\sqrt{x^2 + y^2}} dy \\
= \frac{\ln(4R/\pi) + 1}{\pi} \leq \text{mes}^\tau \ln \frac{8 \text{Re}}{\sqrt{\frac{3}{2}} \text{diam} \tau}.
$$

Substituting this relation into (11), we obtain the assertion of Theorem 2 with the constants $c_1 = c_3 c_4 / \sqrt{\pi}$ and $c_2 = 8 \text{Re} / (\sqrt{3})$; in addition, $\text{diam} \tau \leq \text{diam} \Omega < c_2 < (8 \text{Re} / \sqrt{3}) \text{diam} \Omega$.

**Remark 2.** If the restriction of $\chi \in W^1_2(\Omega)$ onto a triangle $\tau \subset \Omega$ is a polynomial of degree no greater than $r$, then the assertions of the lemma and Theorem 2 remain valid with the constants $c_3 = c_3(r)$ and $c = c(\Omega, r)$. In this case, one can use the “inverse” inequalities from [9] to prove the lemma.

**ACKNOWLEDGMENTS**

This work was supported by the Russian Foundation for Basic Research, project no. 95-01-01421a.

I am grateful to V.B. Andreev for his interest in this work and useful discussions.

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